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แนวเดินแผ่ทั่วที่สั้นที่สุด

On the Length of a Shortest Spanning Walk

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บทคัดย่อ

แนวเดินแผ่ทั่วของกราฟ คือ แนวเดินที่บรรจุจุดยอดทุกจุดในกราฟ จุดประสงค์ของบทความนี้ เพื่อหาจำนวนเส้นเชื่อมของแนวเดินแผ่ทั่วที่สั้นที่สุดในกราฟเชื่อมโยง G ที่มีจุดยอด n จุด ขอบเขตของการศึกษา คือ สนใจจุดยอดเริ่มต้นและสุดท้าย, สนใจจุดยอดเริ่มต้น และไม่สนใจจุดยอดเริ่มต้นและสุดท้าย

ให้ $u,v\in V(G)$ กำหนด $w_2(G)$ คือ จำนวนเส้นเชื่อมของแนวเดินแผ่ทั่วที่สั้นที่สุดที่มีจุดปลายที่ u และ v เมื่อ $u\neq v$, $w_1(G)$ คือ จำนวนเส้นเชื่อมของแนวเดินแผ่ทั่วที่สั้นที่สุดโดยเริ่มต้นที่ v, และ $w_0(G)$ คือ จำนวนเส้นเชื่อมของแนวเดินแผ่ทั่วที่สั้นที่สุด จะเห็นได้ชัดว่า $w_i(G_1)=0$ และ $w_i(G_2)=1$ สำหรับ $i=0,1,2$ เมื่อ G_j คือกราฟเชื่อมโยงที่มีจุดยอด $j=1$ หรือ 2 จุด สำหรับ $n\geq 3$ เราหาขอบเขตแต่ละค่าซึ่งทุกขอบเขตเป็นค่าที่ดีที่สุด และสรุปความสัมพันธ์แต่ละกรณีได้ดังนี้

$n-1\leq w_2(G)\leq 2n-3;$
$n-1\leq w_1(G)\leq 2n-3;$
$n-1\leq w_0(G)\leq 2n-4.$

และสำหรับ $k\in \mathrm{N}$ ที่อยู่ระหว่างค่าขอบเขตบนและขอบเขตล่างของ w_i สำหรับ $i=0,1,2$ จะมีกราฟ G ที่มีจุดยอด n จุด ที่ $w_i(G)=k$

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ABSTRACT

A spanning walk of a graph is a walk that contains all vertices in a graph. The purpose of this article is to find number of edges of a shortest spanning walk of a connected graph G with n vertices. There are 3 cases in the scope of the study: a shortest spanning walk with respect to fixed endpoints; a shortest spanning walk with respect to a fixed initial vertex and a shortest spanning walk without any restriction.

Define $w_2(G) = \max_{u,v \in V(G)}$ {number of edges of a shortest spanning walk with endpoints u and v where $u \neq v\}$, $w_1(G) = \max_{v \in V(G)}$ {number of edges of a shortest spanning walk that starts at $v\}$, and $w_0(G)$ = number of edges of a shortest spanning walk. It is obvious that $w_i(G_1) = 0$ and $w_i(G_2) = 1$ for $i = 0,1,2$ and a connected graph G_j with $j = 1$ or 2 vertices. We find bounds on each parameter in which all bounds are sharp for $n \geq 3$ as follows

$n-1 \leq w_2(G) \leq 2n-3;$
$n-1 \leq w_1(G) \leq 2n-3;$
$n-1 \leq w_0(G) \leq 2n-4.$

Moreover for each k between lower bound and upper bound of any w_i, there exists a graph G with n vertices and $w_i(G) = k$.

คำสำคัญ: แนวเดิน แนวเดินแผ่ทั่ว ทฤษฎีกราฟ

Keywords: Walk, Spanning walk, Graph theory

Introduction

In this paper, we introduce and study a concept of a spanning walk of a graph. We characterize a shortest spanning walk with respect to fixed endpoints, a shortest spanning walk with respect to a fixed initial vertex and a shortest spanning walk without any restriction. The main purpose of this paper is to obtain theorems of number of edges of a shortest spanning walk.

Main Result

A shortest spanning walk with fixed endpoints

Definition: For a connected graph G, let $w_2(G) = \max_{u,v \in V(G)}$ {number of edges of a shortest spanning walk with endpoints u and v where $u \neq v\}$.

Observation: It is obvious that $w_2(G_1)=0$ and $w_2(G_2)=1$

Theorem 1: Let G be a connected graph of order n for $n \ge 3$. Then

(a) $w_2(G) \le 2n-3$ and the bound is sharp;

(b) $w_2(G) \ge n-1$ and the bound is sharp;

(c) If $n-1 \le k \le 2n-3$, then there is a graph H of order n such that $w_2(H)=k$.

Proof: Let $n \ge 3$

(a) We first show that $w_2(G) \le 2n-3$.

Note that we can show inequality by considering only a shortest spanning walk in a spanning tree of a connected graph. Thus we may assume that G is a tree only.

Let $P(n)$ be the statement "if G is a tree of order *n*, then $w_2(G) \le 2n-3$ for all $n \in \mathrm{N}$ such that $n \ge 3$ ".

If $n=3$, then G is a path P_3. So $w_2(G)=3=2 \cdot 3-3$, that is $P(3)$ is true.

Assume that $P(k)$ is true for some $k \ge 3$.

It is well known that any tree of order $n \ge 2$ has $n-1$ edges and at least two vertices of degree 1. Let G be a tree of $k+1$ vertices. Then $E(G)=k$ and there are at least 2 vertices of degree 1. Note that a tree has exactly two vertices with degree 1 if and only if it is a path.

Let $u,v \in V(G)$ such that u is the initial vertex and v is the final vertex of a walk. Let $x \in V(G)$ such that $d(x)=1$. We choose $x \ne u$ and $x \ne v$ if possible.

We want to show that there exists a uv-spanning walk of length at most $2k-1$.

Case 1: $x \ne u$ and $x \ne v$.

Put $G'=G-x$. Then $\left|V(G')\right|=k$. Since $P(k)$ is true, $w_2(G') \le 2k-3$. Thus there is a uv – spanning walk W in G' with the length at most $2k-3$. Adding an edge from W to x and from x to W, we have a uv-spanning walk in G with length at most $2k-1$.

Case 2: $x=u$ or $x=v$.

By the choice of x, the graph G must be a path with endpoints u and v. Thus there exists a uv-spanning walk in G with length k.

Hence, $P(k+1)$ is true.

Therefore by mathematical induction, $P(n)$ is true for all $n \in \mathrm{N}$ such that $n \ge 3$.

To show that the bound is sharp, we illustrate that there is a graph G such that $w_2(G)=2n-3$.

Consider a path P_n as shown in figure 1.

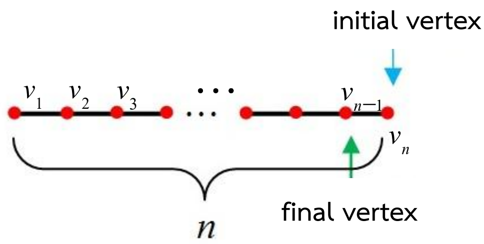

Figure 1.

Notice that a shortest spanning walk from v_n to v_{n-1} in P_n has length $2n-3$. Thus there is a graph G such that $w_2(G)=2n-3$. This completes the proof of (a)

(b) Since every spanning walk of *G* has length at least $n-1$, we have $w_2(G)\geq n-1$.

Consider a complete graph K_n as shown in figure 2.

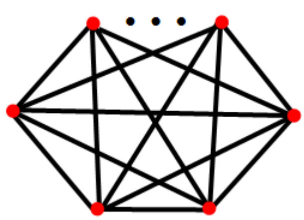

Figure 2.

Note that for any two vertices in K_n, we have a spanning walk of length $n-1$ with those two vertices as endpoints. Hence we have a graph G such that $w_2(G)=n-1$.

(c) Suppose that $n-1\leq k\leq 2n-3$ and $l=k-n+1$.

Consider a graph G as shown in figure 3.

Note that there is a shortest spanning walk with fixed vertices as in figure 3 which has length $n+l-1=n+k-n+1-1=k$.

Thus there exists a graph G of order *n* such that $w_2(G)=k$.

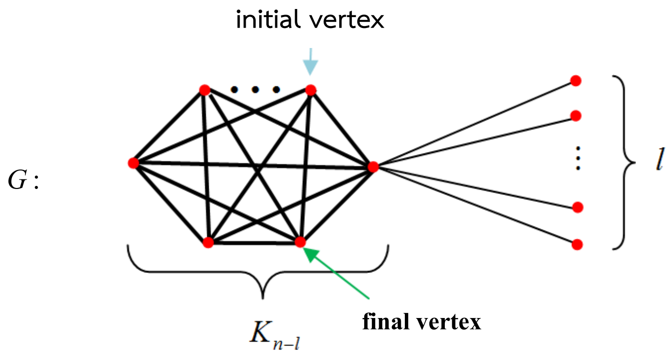

Figure 3.

A shortest spanning walk with fixed initial vertex

Definition: For a connected graph G, let $w_1(G) = \max_{v\in V(G)}$ {number of edges of a shortest spanning walk that starts at v}.

Observation: It is obvious that $w_1(G_1)=0$ and $w_1(G_2)=1$.

Theorem 2: Let G be a connected graph of order n for $n \geq 3$. Then

(a) $w_1(G) \leq 2n-3$ and the bound is sharp;

(b) $w_1(G) \geq n-1$ and the bound is sharp;

(c) If $n-1 \leq k \leq 2n-3$, then there is a graph G of order n such that $w_1(G)=k$.

Proof: Let $n \geq 3$

(a) Since $w_1(G) \leq w_2(G)$, we have $w_1(G) \leq 2n-3$.

Next we show that there is a graph G such that $w_1(G)=2n-3$.

Consider a complete bipartite graph $K_{1,n-1}$ as shown in figure 4.

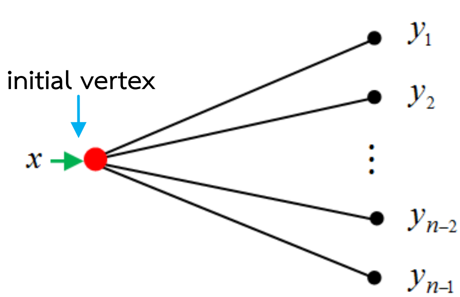

Figure 4.

We see that a walk $xy_1xy_2\ldots xy_{n-2}xy_{n-1}$ is the walk of length $2n-3$ and this is a shortest spanning walk. So we have a graph G such that $w_1(G)=2n-3$.

(b) Since every spanning walk of G has length at least $n-1$, we have $w_1(G)\geq n-1$.

Consider a cycle C_n as shown in figure 5.

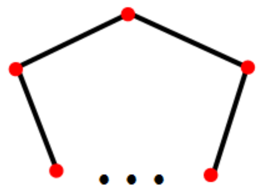

Figure 5.

Note that for every vertex in C_n, We have a spanning walk such that the length of walk is $n-1$.

Hence we have a graph G such that $w_1(G)=n-1$.

(c) Suppose that $n-1\leq k\leq 2n-3$ and $l=k-n+1$.

Consider a graph G as shown in figure 6.

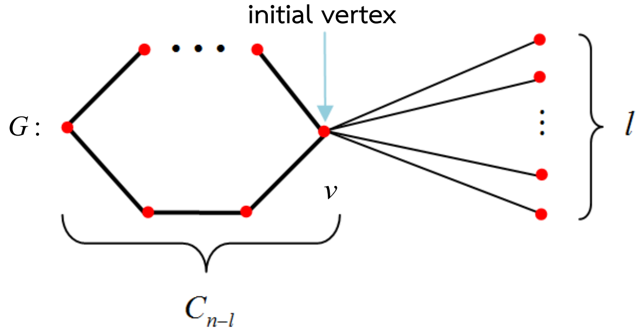

Figure 6.

Note that the length of a spanning walk which starts from the vertex v is at least $n+l-1=n+k-n+1-1=k$. If $x\in V(G)$ and $x\neq v$, then the length of a spanning walk which starts from x is at most $n+l-1=n+k-n+1-1=k$.

Therefore there is a graph G of order n such that $w_1(G)=k$.

A shortest spanning walk

Definition: For a connected graph G, let $w_0(G)$ = number of edges of a shortest spanning walk.

Observation: It is obvious that $w_0(G_1)=0$ and $w_0(G_2)=1$.

Theorem 3: Let G be a connected graph of order n for $n \geq 3$. Then

(a) $w_0(G) \leq 2n-4$ and the bound is sharp;

(b) $w_0(G) \geq n-1$ and the bound is sharp;

(c) If $n-1 \leq k \leq 2n-4$, then there is a graph G of order n such that $w_0(G)=k$.

Proof: Let $n \geq 3$

(a) We first show that there is a graph G such that $w_0(G)=2n-4$

Consider a complete bipartite graph $K_{1,n-1}$ as shown in figure 7.

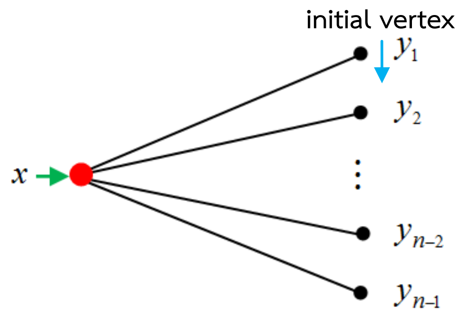

Figure 7.

We see that a walk $y_1xy_2xy_3x\ldots xy_{n-2}xy_{n-1}$ is the walk of length $2n-4$ and this is a shortest spanning walk. So we have a graph G such that $w_0(G)=2n-4$.

We next show that $w_0(G) \leq 2n-4$.

Let G be a graph of order n in figure 8, and let u,v be the initial vertex and final vertex, respectively. Pick $u' \in V(G)$ such that $uu' \in E(G)$.

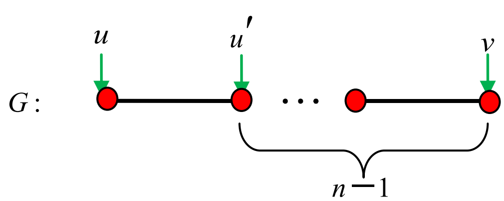

Figure 8.

We now consider $G-u$. Let $G'=G-u$. Note that $|V(G')|=n-1$. Let u' be a neighbor of u in G. Using Theorem 2, we have $w_1(G') \leq 2(n-1)-3=2n-5$. Thus there is a spanning walk W of length at most $2n-5$ with an endpoint u' in G'. Since $uu' \in E(G)$, $W \cup uu'$ is a spanning walk in G of length at most $2n-5+1=2n-4$. Hence, $w_0(G) \le 2n-4$.

This completes the proof of (a)

(b) Since every spanning walk of G has length at least $n-1$, we have $w_0(G) \ge n-1$.

We consider a path P_n as shown in figure 9.

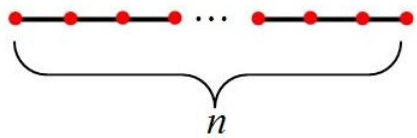

Figure 9.

P_n is a spanning walk of itself and have length $n-1$. Thus the bound is sharp.

(c) Suppose that $n-1 \le k \le 2n-3$ and $l=k-n+1$.

We consider a graph G as shown in figure 10.

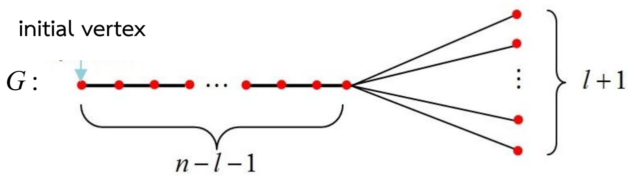

Figure 10.

Note that a shortest spanning walk in G has length $n+l-1=n+k-n+1-1=k$.

Thus there exists a graph G of order n such that $w_0(G)=k$.

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