ปัญหาการระบายสีรายการ

List Coloring Problems

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บทคัดย่อ

การกำหนดค่ารายการ L ของกราฟ G คือฟังก์ชันที่กำหนดเซตของสีให้จุดยอดทุกจุดของกราฟและ G จะเรียกว่า ระบายสีได้แบบ-L ถ้าจุดยอด v แต่ละจุดสามารถระบายสีได้โดยใช้สีจาก $L(v)$ และแต่ละจุดยอดที่อยู่ติดกันมีสีต่างกัน

ในบทความวิชาการนี้ เริ่มต้นโดยการแนะนำนิยามที่เกี่ยวกับปัญหาการกำหนดค่ารายการ หลังจากนั้นจะกล่าวถึงงานวิจัยเกี่ยวกับปัญหาการกำหนดค่ารายการ เช่น ลักษณะเฉพาะของกราฟเลือกได้แบบ-2, ลักษณะเฉพาะของกราฟเลือกได้แบบ-3 ของกราฟสองส่วนแบบบริบูรณ์ และ การกำหนดค่ารายการที่ทำให้ $K_{7,7}$ $K_{7,8}$ ไม่เป็นกราฟเลือกได้แบบ-3 สุดท้ายจะพูดถึงหัวข้อที่อาจจะเป็นงานวิจัยได้

ABSTRACT

A *list assignment* L of a graph G is a function which assigns a set of colors to all vertices and G is called L-colorable if each vertex v can be colored by using color from $L(v)$ and adjacent vertices receive distinct colors.

In this article, definitions related to list assignment problems are introduced. Then some research of list assignment problems is proposed; for example, a characterization of 2-choosable graphs, a characterization of 3-choosable complete bipartite graphs and a list assignment such that prevents $K_{7,7}$ and $K_{7,8}$ from being 3-choosable. Finally, topics of possible future research are presented.

คำสำคัญ: การกำหนดค่ารายการ การระบายสี รงค์-เลือกได้ กราฟสองส่วน กราฟเชิงระนาบ

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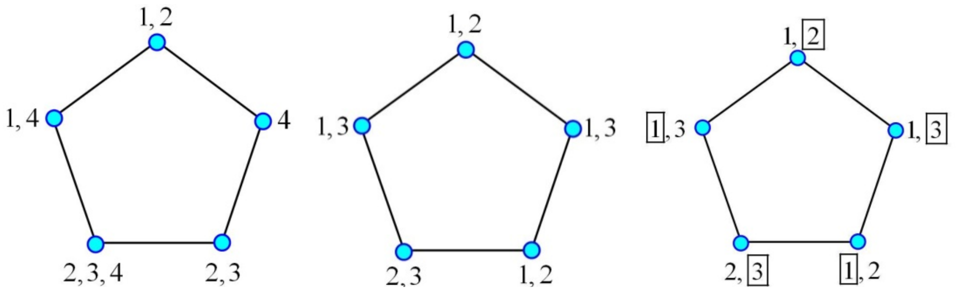
1. Definitions

A *list assignment* of a graph *G* is a function which assigns a set of colors, called a *list* to all vertices. A list assignment *L* of *G* is a *k-list assignment* if $|L(v)| = k$ for all $v \in V(G)$. For a list assignment *L* of *G*, a coloring *f* of *G* is an *L-coloring* of *G* if *f*(*v*) is chosen from *L*(*v*) for each vertex $v \in V(G)$ and adjacent vertices receive distinct colors. A graph *is L-colorable* if it has an *L*-coloring. For convenience, we usually write a list of each vertex without any bracket. For example, the list assignments of C_5 are shown in Fig. 1.



A list assignment A 2-list assignment, say *L* An *L*-coloring

Figure 1. list assignments of C_5

A graph *G* is *k*-choosable if it is *L*-colorable for every *k*-list assignment *L* of *G.* The smallest positive integer *k* satisfying this property is called the *list chromatic number* of *G*, denoted by $\chi_l(G)$.

To prove a graph is *k*-choosable, we need to characterize all *k*-list assignments *L* and prove that it is *L*-colorable. Then this problem is complicated. For example, the proof of C_5 is 3-choosable is shown in Fig. 2.

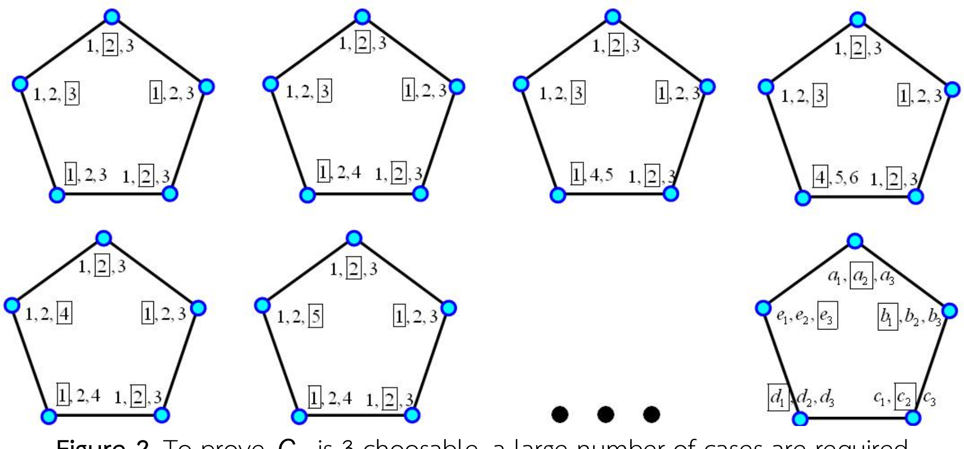


Figure 2. To prove C_5 is 3-choosable, a large number of cases are required.

2. Literature Review

The problem of list assignments is first published independently by (Vizing, 1976) and (Erdos et al., 1979). The authors gave a characterization of 2-choosable graphs. They stated that a graph *G* is 2-choosable if and only if the core of *G* belongs to $\{K_1, C_{2m+2}, \theta_{2,2,2m}; m \geq 1\}$.

To clearify the statement, new definitions are related; core, cycle and theta graph. The *core* of a graph *G* is obtained from successively removing a vertex with degree 1 from *G*. For example, the core of some graph is shown in Fig. 3.

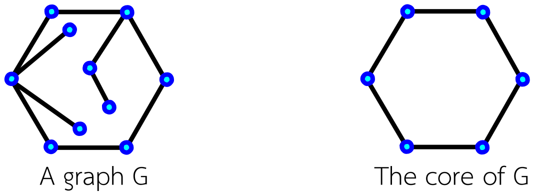


Figure 3. A graph and its core

A *complete graph* is a graph whose vertices are pairwise adjacent; the complete graph with *n* vertices is denoted by K_n. The following figures present complete graph K_n for *n* = 3, 4, 5 and 6

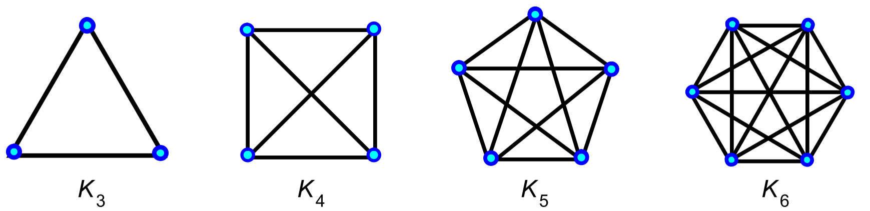


Figure 4. Examples of complete graphs

A *cycle* is a graph with an equal number of vertices and edges whose vertices can be placed around a circle so that two vertices are adjacent if and only if they appear consecutively along the circle; the cycle with *n* vertices is denoted by C_n. The examples of cycle C_n are shown in Fig. 5.

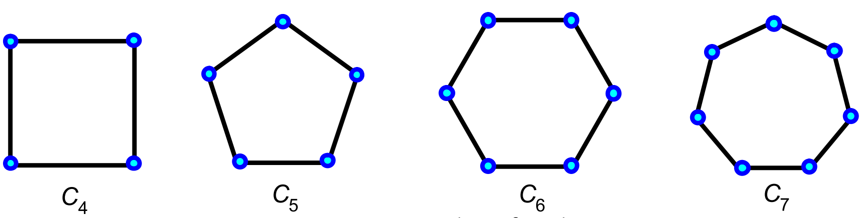


Figure 5. Examples of cycles

The ***theta graph*** $\theta_{a,b,c}$ is the graph consisting of three internally disjoint paths with common endpoints and lengths *a*, *b* and *c* with $a \leq b \leq c$. For example, the theta graph $\theta_{a,b,c}$ where (*a*,*b*,*c*) = (1,2,2), (2,2,2) และ (2,2,3)

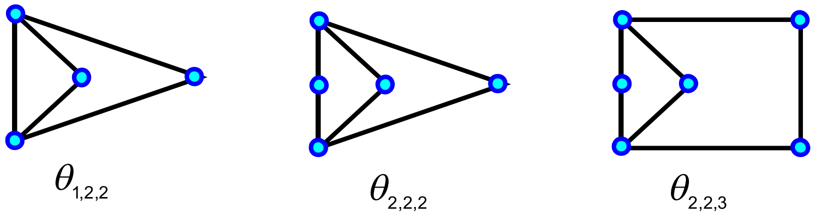


Figure 6. Examples of theta graphs

However, there is no characterization of *k*-choosable graphs for $k \geq 3$. Given a positive integer *k*, *k*-choosable graphs are studied only for some classes of graphs; for example, bipartite graphs and planar graphs.

A *bipartite* graph is one whose vertex set can be partitioned into 2 subsets *X* and *Y*, so that each edge has an endpoint in *X* and another endpoint in *Y*; subsets *X* and *Y* are called *partite sets.* A *complete bipartite graph,* is a bipartite graph with partitions *X* and *Y* in which each vertex of *X* is adjacent to each vertex in *Y*; if |*X*| = *a* and |*Y*|=*b*, such a graph is denoted by $K_{a,b}$. The following figures are complete bipartite graphs $K_{a,b}$ where (*a*,*b*) = (2,3), (2,4) and (3,3).

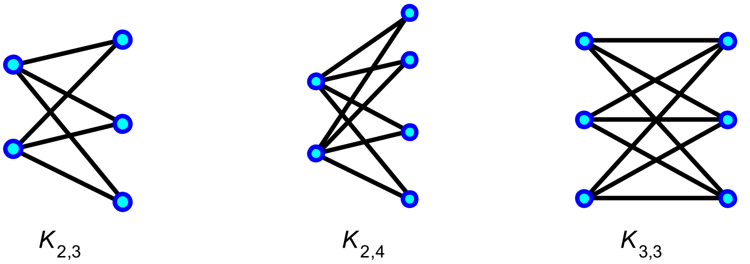


Figure 7. Examples of complete bipartite graphs

For $3 \leq m \leq n$, $K_{m,n}$ is 3-choosable if and only if $m = 3$ and $n \leq 26$ (Erdos et al., 1979), or $m = 4$ and $n \leq 20$ (Mahadev et al., 1991), or $m = 5$ and $n \leq 16$ (Shende and Tesman, 1994), or $m = 6$ and $n \leq 10$ (O'Donnell, 1995). When $7 \leq m, n$, $K_{m,n}$ is not 3-choosable because $K_{7,7}$ is not 3-choosable (Erdos et al., 1979). Fig. 8 shows a 3-list assignment *L* such that $K_{7,7}$ is not *L*-colorable.

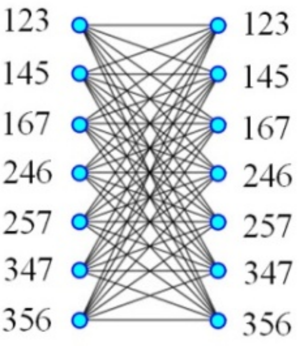


Figure 8. A 3-list assignment that prevents $K_{7,7}$ from being 3-choosable, say L_F

In 1996, Hanson et al. (1996) stated that every complete bipartite graph with 13 vertices is 3-choosable. Later, Fitzpatrics and MacGillivray, (2005) added that every complete bipartite graph with 14 vertices except $K_{7,7}$ is 3-choosable. Furthermore, there is a unique 3-list assignment, say L_F, up to renaming the colors such that $K_{7,7}$ is not L_F-colorable. Recently, Charoenpanitseri et al., (2013) proved that every complete bipartite graph with 15 vertices except $K_{7,8}$ is 3-choosable and a 3-list assignment L of $K_{7,8}$ is a non-colorable list assignment if and only if $L|_{V(K_{7,7})} = L_F$

A graph is *planar* if it has a drawing without crossings. The following figures show the difference between planar graphs and non-planar graphs.

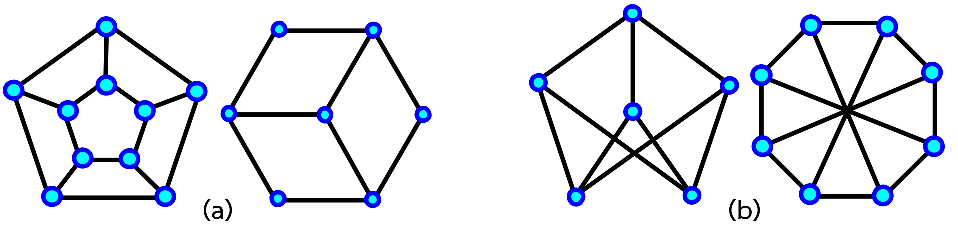


Figure 9. Examples of (a) planar graphs and (b) non-planar graphs

In 1994, (Thomassen, 1994) proved that all planar graphs are 5-choosable while some planar graphs are 3-choosable. (See Lam et al., 2005; Thomassen, 1995; Zhang, 2005; Zhang and Xu, 2004; Zhang et al., 2006; Zhu et al., 2007 for more detail.)

3. How to color a graph.

In this section, Example 1 and Example 3 show how to prove graphs are L-colorable and Example 2 shows how to prove a graph is not L-colorable.

Example 1. Let L be a 2-list assignment of a graph G as shown in Fig. 10.

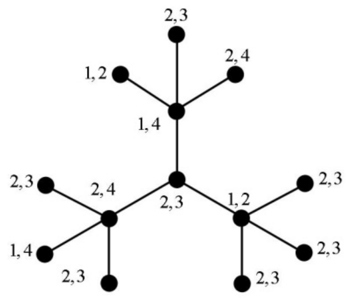


Figure 10. A graph that we need to color.

There are several ways to show that *G* is *L*-colorable. For example, start labeling a vertex with degree 4 by color 1 as shown in Fig. 11 (a). Then label 3 vertices with degree 1 that are adjacent to the labeled vertex as shown in Fig. 11 (b).

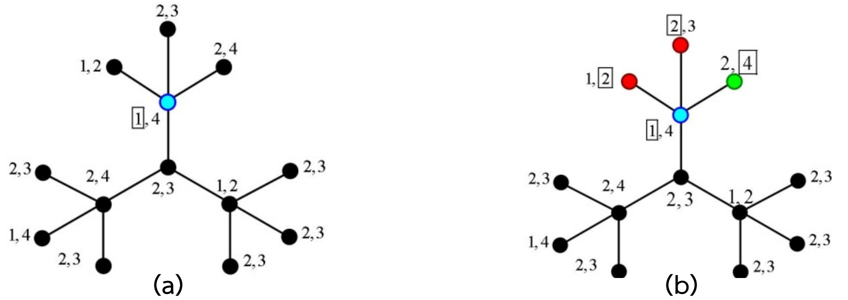


Figure 11. Start labeling some vertices

Then we continue labeling a vertex that is adjacent to any labeled vertices. Finally, *G* is completely labeled as shown in Fig. 12.

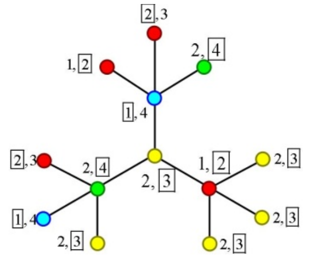


Figure 12. An *L*-colorable of *G*

Example 2. Let *L* be a 2-list assignment of a graph *G* as shown in Fig. 13.

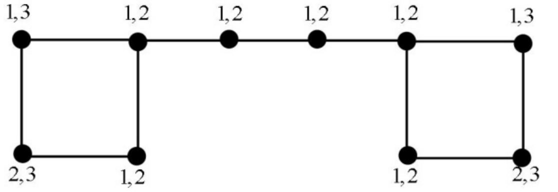


Figure 13. A graph that we need to color

We will show that G is not L-colorable. If a vertex with degree 3 is labeled by color 1, then a vertex cannot be labeled. (See Fig. 14.)

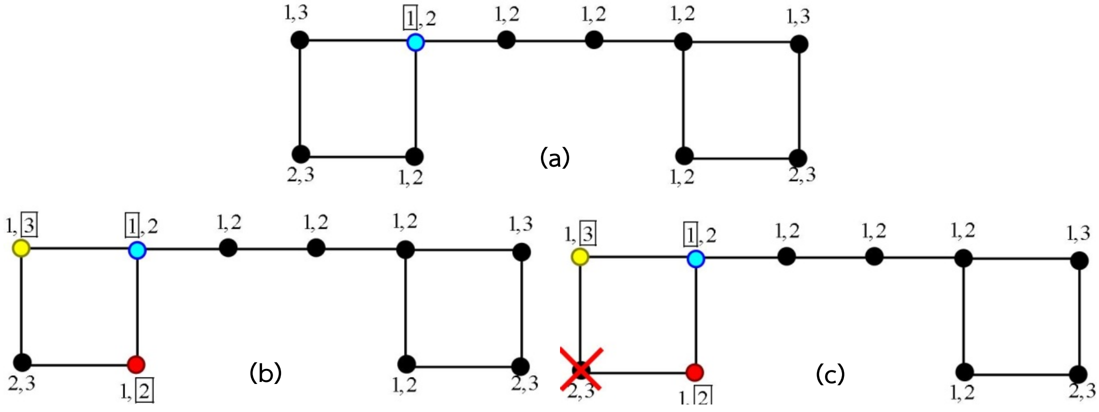


Figure 14. A vertex with degree 3 cannot be labeled by color 1

Hence, both vertices with degree 3 must be labeled by color 2. Then a vertex between two vertices with degree 3 cannot be labeled. (See Fig. 15.)

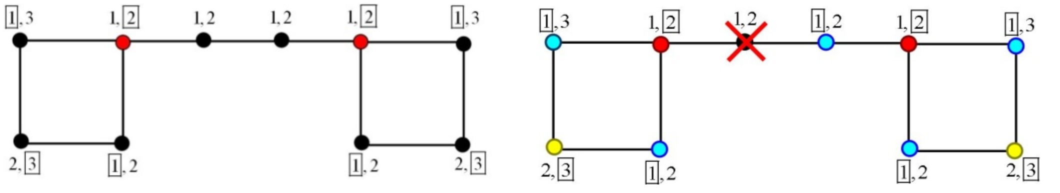


Figure 15. G is not L-colorable.

Example 3. Let C_6 be the graph shown in Fig. 16.

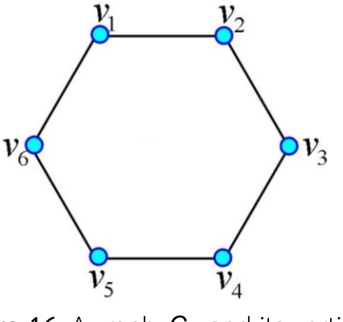


Figure 16. A graph C_6 and its vertices

Consider the number of all possible 2-list assignments L. Since C_6 have 6 vertices and each vertex has 2 available colors, there are at most 12 colors. Then the number of 2-list assignments with at most 12 colors is $\binom{12}{2}^6 = 82,653,950,016$. It is impossible to label all list assignments one by one. However, we group cases together and prove that all cases can be labeled.

Case 1. Each vertex has the same 2 colors, say color 1 and color 2. Then we label v_1, v_3, v_5 by color 1 and label v_2, v_4, v_6 by color 2.

Case 2. Some vertices have different colors. Suppose that $L(v_1) \neq L(v_6)$ and $c_1 \in L(v_1) \setminus L(v_6)$. We label v_1 by color c_1. Let $c_2 \in L(v_2) \setminus \{c_1\}$ and we label v_2 by color c_2 and so on. Notice that the key in this case is that color c_1 is not in $L(v_6)$. After v_1 is labeled, the vertex v_6 still has 2 available colors.

4. Conclusions

Many problems related to list assignments are still open. Hence, we will talk about topics of possible research. According to Section 2, there is no characterization of 3-choosable graphs. Finding a necessary or a sufficient condition of 3-choosable graphs can become a topic. Finding a class of graphs which is 3-choosable can become a topic, too. Although $K_{7,9}$ and $K_{8,8}$ are not 3-choosable because of containing $K_{7,7}$ which is not 3-choosable, we may investigate all 3-list assignments that prevent the graphs from being 3-choosable. Moreover, we may investigate values of m, n such that $K_{m,n} - e$ is 3-choosable when *e* is an edge of $K_{m,n}$.

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