การทำสมการเชิงอนุพันธ์สามัญอันดับสี่ให้เป็นเชิงเส้น โดยการแปลงของซันด์แมนแบบวางนัยทั่วไป

Linearization of Fourth-order Ordinary Differential Equations by Generalized Sundman Transformations

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บทคัดย่อ

งานวิจัยนี้ศึกษาปัญหาการทำให้เป็นเชิงเส้นสำหรับสมการเชิงอนุพันธ์สามัญอันดับสี่โดยใช้การแปลงของซันด์แมนแบบวางนัยทั่วไป กล่าวคือ

$$u = F(x,y)$$
$$dt = G(x,y)dx$$

จากการศึกษา ทำให้พบเงื่อนไขจำเป็นและเงื่อนไขเพียงพอสำหรับสมการเชิงอนุพันธ์สามัญอันดับสี่ที่สมมูลกับรูปทั่วไปของสมการเชิงอนุพันธ์สามัญเชิงเส้นอันดับสี่ งานวิจัยนี้ได้ผลลัพธ์ที่สมบูรณ์สำหรับกรณี $F_x = 0$ นอกจากนี้ยังได้แสดงตัวอย่างการประยุกต์ใช้สำหรับสมการเชิงอนุพันธ์ย่อยไม่เชิงเส้นอันดับสี่

$$u_{tt} = (\tilde{\kappa} u + \tilde{\gamma} u^2)_{xx} + \tilde{\nu} u u_{xxxx} + \tilde{\mu} u_{xxtt} + \tilde{\alpha} u_x u_{xxx} + \tilde{\beta} u_{xx}^2,$$

เมื่อ $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\mu}, \tilde{\nu},$ และ $\tilde{\kappa}$ เป็นค่าคงตัว

ABSTRACT

The linearization problem of fourth-order ordinary differential equations by the generalized Sundman transformation, i.e.,

$$u = F(x,y),$$
$$dt = G(x,y)dx,$$

is considered in the paper. Necessary and sufficient conditions for fourth-order ordinary differential equations to be linearizable into the general form of a linear fourth-order ordinary differential equation are obtained. Here, a complete solution is given for the case $F_x = 0$. We also give an example which apply our procedure for a nonlinear fourth-order partial differential equation

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where $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\mu}, \tilde{\nu},$ and $\tilde{\kappa}$ are arbitrary constants.

คำสำคัญ: ปัญหาการทำให้เป็นเชิงเส้น การแปลงของซันด์แมนแบบวางนัยทั่วไป สมการเชิงอนุพันธ์สามัญอันดับสี่
Keywords: Linearization problem, Generalized Sundman transformations,
Fourth-order ordinary differential equations

INTRODUCTION

Differential equations are used as an important tool to solve problems of sciences and physical phenomena. In general, these equations are very difficult to find solutions. One of the fundamental methods for solving them makes use of a change of variables that transform a given differential equation into another differential equation with known properties. Since the class of linear equations is considered to be the simplest class of equations, there arises the problem of transforming given differential equations into linear equations. This problem is called the *linearization problem*, which is a particular case of an equivalence problem. Transformations used for solving a linearization problem are point transformations, contact transformations, tangent transformations, and generalized Sundman transformations.

The linearization problem has been studied in many publications. First of all, Lie (1883) studied the problem of linearizing a second-order ordinary differential equation by point transformations. He found that any linearizable second-order ordinary differential equation can be at most cubic in the first-order derivative, and provided a linearization test in terms of its coefficients. Grebot (1997) studied the linearization of third-order ordinary differential equations by means of a restricted class of point transformations, namely, $t = \varphi(x),\ u = \psi(x, y)$. However, the problem was not completely solved. Complete criteria for linearization by means of point transformations were obtained by Ibragimov and Meleshko (2005). The linearization of fourth-order ordinary differential equations by point transformations was discussed by Ibragimov et al. (2007).

Lie also noted that all second-order ordinary differential equations can be transformed into the trivial equation $u'' = 0$ by means of contact transformations, but this is not the case for higher-order ordinary differential equations. Therefore, the linearization problem using contact transformations becomes interesting for ordinary differential equations of order greater than two. Linearization of third-order ordinary differential equations with respect to contact transformations was studied by Neut and Petitot (2002). Ibragimov and Meleshko (2005) presented the explicit form of the linearization criteria. Dridi and Neut (2005) solved a particular linearization problem for a fourth-order ordinary differential equation. They found conditions for a fourth-order ordinary differential equation to be equivalent to $u^{(4)} = 0$ under contact transformations. Complete criteria for fourth-order ordinary differential equations to be linearizable via contact transformations were given by Suksern et al. (2009).

Tangent transformations were applied for the linearization problem of fourth-order ordinary differential equations. Complete study of fiber preserving transformations $(\varphi_p = \psi_y = 0)$ mapping fourth-order ordinary differential equations to trivial third-order ordinary differential equation $y''' = 0$ was given in Suksern and Meleshko (2014). Nakpim (2016) found necessary and sufficient conditions for third-order ordinary differential equations to be equivalent to the Laguerre form of a linear second-order ordinary differential equation $u'' = 0$.

The generalized Sundman transformation was earlier considered for second-order ordinary differential equations by Duarte et al. (1994) using the Laguerre form. Nakpim and Meleshko (2010a) gave examples which show that the Laguerre form is not sufficient for the linearization problem via generalized Sundman transformations. Criteria for a third-order ordinary differential equation to be equivalent to the linear equation

$$u''' = 0$$

with respect to the generalized Sundman transformation were presented in Euler et al. (2003). Nakpim and Meleshko (2010b) obtained necessary and sufficient conditions for a third-order ordinary differential equation to be linearizable into $u''' + \alpha u = 0,$ where α is a constant. Some applications of generalized Sundman transformations to ordinary differential equations were considered in Berkovich (2001) and earlier papers, which were summarized in Berkovich (2002). Suksern and Tummakun (2014) considered the linearization problem for nonlinear fourth-order ordinary differential equations to be equivalent to the trivial equation $u^{(4)} = 0$ by the generalized Sundman transformation.

According to the Laguerre theorem (1879), in any linear ordinary differential equations, the two terms of second-highest order can be simultaneously removed by point transformations. Thus, the Laguerre form of a linear ordinary differential equation is

$$y^{(n)} + a_{n-3}(x)y^{(n-3)} + \cdots + a_1(x)y' + a_0(x)y = 0.$$

Although the composition of point transformations is a point transformation, this is not the case for generalized Sundman transformations: the composition of a point transformation and a generalized Sundman transformation is not necessarily a generalized Sundman transformation. Hence, for the linearization problem via generalized Sundman transformations, we need to use the general form of a linear ordinary differential equation instead of the Laguerre form.

The solution of the linearization problem via the generalized Sundman transformation of fourth-order ordinary differential equations given in Suksern and and Tummakun (2014) only gives particular criteria for linearizable equation. In this paper, we find necessary and sufficient conditions which allow a fourth-order ordinary differential equation $y^{(4)} = H(x, y, y', y'', y''')$ to be transformed into a linear equation $u^{(4)} + \alpha u''' + \beta u'' + \gamma u' + \eta u = \delta$ where $\alpha,\ \beta,\ \gamma,\ \eta,$ and δ are arbitrary constants via the generalized Sundman transformation. Complete analysis of the compatibility of arising equations is given for the case $F_x = 0.$

GENERALIZED SUNDMAN TRANSFORMATIONS

A generalized Sundman transformation is a non-point transformation defined by the formulae

$$u(t) = F(x, y), \quad dt = G(x, y)dx, \quad F_y G \neq 0. \qquad (1)$$

Let us explain how the generalized Sundman transformation maps one function into another. Assume that $y_0(x)$ is a given function of x. Integrating the second equation of (1), we obtain

$$t = Q(x)$$

for some function $Q(x).$ Using the inverse function theorem, we find that $x = Q^{-1}(t).$ Substituting x into the function $F(x, y_0(x))$, we obtain the transformed function

$$u_0(t) = F\left(Q^{-1}(t), y_0\big(Q^{-1}(t)\big)\right).$$

Conversely, let $u_0(t)$ be a given function of $t.$ Using the inverse function theorem, we solve the equation

$$u_0(t) = F(x, y)$$

with respect to y, where $y = \phi(x,t)$ for some function $\phi(x,t)$. Solving the ordinary differential equation

$$\frac{dt}{dx} = G\big(x, \phi(x,t)\big),$$

we find that $t = H(x)$ for some function $H(x)$. The function $H(x)$ can be written as an action of a functional $H = \mathcal{L}(u_0)$. Substituting $t = H(x)$ into the function $\phi(x,t)$, the transformed function $y_0(x) = \phi\big(x, H(x)\big)$ is obtained.

Notice that for the case $G_y = 0$, the generalized Sundman transformation becomes a point transformation. Hence, we shall assume from now on that $G_y \neq 0$.

NECESSARY CONDITIONS FOR LINEARIZATION

We start by obtaining necessary conditions for the linearization problem. First, we find the general form of fourth-order ordinary differential equation

$$y^{(4)} = H(x, y, y', y'', y'''),$$

which can be mapped via a generalized Sundman transformation

$$u(t) = F(x,y),\ dt = G(x,y)dx, \tag{2}$$

into the general form of linear equation

$$u^{(4)} + \alpha u''' + \beta u'' + \gamma u' + \eta u = \delta, \tag{3}$$

where α, β, γ, η, and δ are constants. The independent variable t is defined by the functional $\mathcal{L}(u)$. If $G_y \neq 0$ and the coefficients of (3) are not constants, then the substitution of t into (3) gives a functional equation.

The function u and its derivatives u', u'', u''', and $u^{(4)}$ are defined by the first formula (2) and its derivatives with respect to t, i. e.,

$$u' = \frac{1}{G}(F_x + F_y y'),$$

$$u'' = \frac{1}{G^3}\Big(2F_{xy}Gy' + F_{xx}G + F_{yy}Gy'^2 - F_x\big(G_x + G_y y'\big) + F_y\big(Gy'' - G_y y'^2 - G_x y'\big)\Big),$$

$$\begin{aligned}
u''' = \frac{1}{G^5}[&G^2\big(F_{yyy}y'^3 + 3F_{xyy}y'^2 + 3F_{xxy}y' + F_{xxx}\big) - 3F_{xx}G\big(G_y y' + G_x\big) + 3F_{xy}G(Gy'' - 2G_y y'^2) \\
&+3F_{yy}Gy'\big(G\,y'' - G_y y'^2 - G_x y'\big) + (F_y y' + F_x)(-G_{xx}G - 2G_{xy}Gy' - G_{yy}Gy'^2) \\
&+3F_x(G_y^2 y'^2 + G_x^2) - F_x G_y G(y'' - 6G_x y') - 3F_y G_x(Gy'' - 2G_y y'^2 - G_x y') - F_y G_y y'(4Gy'' \\
&-3G_y y'^2) + F_y G^2 y'''],
\end{aligned}$$

$$\begin{aligned}
u^{(4)} = \frac{1}{G^7}[&G^3(F_{yyyy}y'^4 + 4F_{xyyy}y'^3 + 6F_{xxyy}y'^2 + 4F_{xxxy}y' + F_{xxxx}) + (G_y y' + G_x)(-6F_{xxx}G^2 \\
&+10F_y G_{yy}Gy'^3 + 20F_y G_{xy}Gy'^2 + 10F_x G_{yy}Gy'^2 - 45F_y G_x G_y y'^2 + 10F_y G_{xx}Gy' \\
&+20F_x G_{xy}Gy' - 45F_x G_x G_y y' + 10F_x G_{xx}G) + 6F_{xxy}G^2(Gy'' - 3G_y y'^2 - 3G_x y') \\
&+6F_{xyy}G^2 y'(2Gy'' - 3G_y y'^2 - 3G_x y') + 6F_{yyy}Gy'^2(G^2 y'' - G_y y'^2 - G_x Gy') + (F_y y' \\
&+F_x)(-G_{xxx}G^2 - 3G_{xxy}G^2 y' - 3G_{xyy}G^2 y'^2 - G_{yyy}G^2 y'^3) - F_{xx}G^2(3F_x G_{xy} + 4G_y y'' \\
&+4G_{yy}y'^2 + 8G_{xy}y' + 4G_{xx}) + 15F_{xx}G(G_y^2 y'^2 + 2G_x G_y y' + G_x^2) + 4F_{xy}G^2(Gy''' \\
&-2G_{yy}y'^3 - 4G_{xy}y'^2 - 2G_{xx}y') + 30F_{xy}Gy'(G_y^2 y'^2 + 2G_x G_y y' + G_x^2) - F_{xy}G^2 y''(26G_y y' \\
&+18G_x) + F_{yy}G^2(4Gy'y''' + 3Gy''^2 - 22G_y y'^2 y'' - 18G_x y'y'' - 4G_{yy}y'^4 - 8G_{xy}y'^3 \\
&-4G_{xx}y'\text{^}2) + 15F_{yy}G(G_y^2 y'^4 + 2G_x G_y y'^3 + G_x^2 y'^2) + 5F_x G_y^2 y'(2Gy'' - 3G_y y'^2) \\
&-F_x G\big(G_y Gy''' - 10G_x G_y y'' + 3G_{yy}Gy'y''\big) - 15G_x^3\big(F_y y' + F_x\big) - 4F_y G^2 y''^2\big(G_y + G_{xx}\big) \\
&-F_y Gy'y''(11GG_{xy} - 40G_x G_y) + F_y G_x G(15G_x y'' - 6Gy''') - 7F_y G^2 y'(G_y y''' + G_{yy}y'y'') \\
&+5F_y G_y^2 y'^2(5Gy'' - 3G_y y'^2) + F_y G^3 y^{(4)}].
\end{aligned}$$

Substituting u', u'', u''', and $u^{(4)}$ into (3) with constant coefficients, we obtain equation

$$\begin{aligned}
&y^{(4)} + \lambda_0(x,y)y'y''' + \lambda_1(x,y)y''' + \lambda_2(x,y)y'^2 y'' + \lambda_3(x,y)y'y'' + \lambda_4(x,y)y''^2 + \lambda_5(x,y)y'' \\
&+ \lambda_6(x,y)y'^4 + \lambda_7(x,y)y'^3 + \lambda_8(x,y)y'^2 + \lambda_9(x,y)y' + \lambda_{10}(x,y) = 0,
\end{aligned} \tag{4}$$

where the coefficients $\lambda_i(x,y)$ $(i = 0, 1, 2, \dots, 10)$ are related to the functions F and G in the following way:

$\lambda_0 \quad =(4F_{yy}G - 7F_yG_y)/(F_yG),$ (5)

$\lambda_1 \quad =(4F_{xy}G - F_xG_y - 6F_yG_x + F_y\,\alpha G^2)/(F_yG),$ (6)

$\lambda_2 \quad =(6F_{yyy}G^2 - 22F_{yy}G_yG - 7F_yG_{yy}G + 25F_yG_y^2)/(F_yG^2),$ (7)

$\lambda_3 \quad =(12F_{xyy}G^2 - 26F_{xy}G_yG - 3F_xG_{yy}G - 3F_{yy}G(6G_x - \alpha G^2) - 11F_yG_{xy}G + 10F_xG_y^2$
$+4F_yG_y(10G_x - \alpha G^2)/(F_yG^2),$ (8)

$\lambda_4 \quad =(3F_{yy}G - 4F_yG_y)/(F_yG),$ (9)

$\lambda_5 \quad =(6F_{xxy}G^2 - 4F_{xx}G_yG - 3F_{xy}G(6G_x - \alpha G^2) - 3F_xG_{xy}G - 4F_yG_{xx}G + F_xG_y(10G_x - \alpha G^2)$
$+3F_yG_x(5G_x - \alpha G^2) + F_y\beta G^4)/(F_yG^2),$ (10)

$\lambda_6 \quad =(F_{yyyy}G^3 - 6F_{yyy}G_yG^2 - F_yG_{yyy}G^2 - 4F_{yy}G_{yy}G^2 + 15F_{yy}G_y^2G + 5F_yG_y(2G_{yy}G$
$-3G_y^2))/(F_yG^3),$ (11)

$\lambda_7 \quad =(4F_{xyyy}G^3 - 18F_{xyy}G_yG^2 - F_{yyy}G^2(6G_x + \alpha G^2) - 3F_yG_{xyy}G^2 - F_xG_{yyy}G^2 + 30F_{xy}G_y^2G$
$-G_{yy}G\left(8F_{xy}G - 10F_xG_y - F_y\alpha G^2\right) + 3F_{yy}G_yG(10G_x - \alpha G^2) - 4G_{xy}G\left(2F_{yy}G - 5F_yG_y\right)$
$+ 2F_yG_x(5G_{yy}G - 2G_y^2) - 3G_y^3(5F_x - F_y\alpha G^2))/(F_yG^3),$ (12)

$\lambda_8 \quad =(6F_{xxyy}G^3 - 18F_{xxy}G_yG^2 - 3F_{xyy}G^2(6G_x - \alpha G^2) - 3F_yG_{xxy}G^2 - 3F_xG_{xyy}G^2$
$-2F_{xy}G\left(8G_{xy}G - 30G_xG_y + 3G_y\alpha G^2\right) - F_{xx}G\left(4G_{yy}G - 15G_y^2\right) + F_xG_{yy}G(10G_x - \alpha G^2)$
$+20G_{xy}G\left(F_xG_y + F_yG_x\right) - 3F_xG_y^2(15G_x - \alpha G^2) - 2F_yG_{xy}\alpha G^3 + 3F_{yy}G_xG\,(5G_x - \alpha G^2)$
$-F_{yy}G^2(4G_{xx} - \beta G^3) + 10F_yG_{xx}G_yG - F_yG_y(45G_x^2 - 6G_x\alpha G^2 + \beta G^4))/(F_yG^3),$ (13)

$\lambda_9 \quad =(4F_{xxxy}G^3 - 6F_{xxx}G_yG^2 - 3F_{xxy}G^2(6G_x - \alpha G^2) - F_yG_{xxx}G^2 - 3F_xG_{xxy}G^2 - 8F_{xx}G_{xy}G^2$
$-8F_{xy}G_{xx}G^2 + 2F_{xy}\beta G^5 + (10G_x - \alpha G^2)\left(3F_{xx}G_yG + 2F_xG_{xy}G + F_yG_{xx}\right) + 10F_xG_{xx}G_yG$
$-45F_xG_x^2G_y + 6F_xG_xG_y\alpha G^2 + \left(6F_{xy}G_xG - 3F_yG_x^2\right)(5G_x - \alpha G^2) - \beta G^4\left(F_xG_y + F_yG_x\right)$
$+F_yG^6\gamma)/(F_yG^3),$ (14)

$\lambda_{10} \quad =(F_{xxxx}G^3 - F_{xxx}G^2(6G_x - \alpha G^2) - F_xG_{xxx}G^2 - 4F_{xx}G_{xx}G^2$
$+(\alpha G^2 + 5G_x)(-3F_{xx}G_xG - 3F_xG_x^2) + F_{xx}\beta G^5 - F_xG_{xx}\alpha G^3 + 10F_xG_{xx}G_xG - F_xG_x\beta G^4$
$+F_xG^6\gamma - \delta G^7 + \eta FG^7)/(F_yG^3),$ (15)

where $F_yG \neq 0$.

The necessary form of a fourth-order ordinary differential equation which can be mapped into a linear fourth-order ordinary differential equation (3) via the generalized Sundman transformation (2) is presented by equation (4).

SUFFICIENT CONDITIONS FOR LINEARIZATION

To obtain sufficient conditions, we have to solve the compatibility problem of the system (5)-(15) by considering these equations as an overdetermined system of partial differential equations, where coefficients $\lambda_i(x,y)(i = 0,1,2,\ldots,10)$ are known functions, and F, G are unknown functions with the independent variables x, y. Further analysis of the compatibility depends on the quantity of F_x. Thus, for linearization problem, we need to study two cases: $F_x = 0$ and $F_x \neq 0$. Here, a complete solution for the case $F_x = 0$ is given.

Solving equation (5)-(7), (9)-(11), and (14)-(15), we obtain

$F_{yy} \quad =(F_y(-4\lambda_0 + 7\lambda_4))/5,$ (16)

$\alpha \quad =(6G_x + G\lambda_1)/G^2,$ (17)

$\lambda_{0y} \quad =(70\lambda_{4y} - 6\lambda_0^2 + 46\lambda_0\lambda_4 - 25\lambda_2 - 34\lambda_4^2)/15,$ (18)

$G_y \quad =(G(-3\lambda_0 + 4\lambda_4))/5,$ (19)

$\beta \quad =33G_x^2 + 11G_xG\lambda_1 - 2\lambda_{1x}G^2 + 3G^2\lambda_5,$ (20)

λ_{4yy} $=(15\lambda_{2y}-30\lambda_{4y}\lambda_0-25\lambda_{4y}\lambda_4-9\lambda_0^2\lambda_4+15\lambda_0\lambda_2-16\lambda_0\lambda_4^2+10\lambda_2\lambda_4+19\lambda_4^2$
$-45\lambda_6)/15,$ (21)

γ $=(216G_x^2+108G_x^2G\lambda_1-24G_x\lambda_{1x}G^2+7G_xG^2\lambda_1+36G_xG^2\lambda_5-6\lambda_{1xx}G^3-7\lambda_{1x}G^3\lambda_1$
$+36G^3\lambda_9)/36G^6,$ (22)

δ $=(-F_y\lambda_{10}+\eta FG^4)/G^4.$ (23)

Comparing the mixed derivatives $(F_x)_{yy}=(F_{yy})_x$ and differentiating $\alpha,\beta,\gamma,$ and δ with respect to x and y, we obtain the following equations:

λ_{0x} $=(7\lambda_{4x})/4,$ (24)

G_{xx} $=(12G_x^2+G_xG\lambda_1-\lambda_{1x}G^2)/6G,$ (25)

$36G_x\lambda_0-48G_x\lambda_4+10\lambda_{1y}G-15\lambda_{4x}G+6G\lambda_0\lambda_1-8G\lambda_1\lambda_4=0,$ (26)

$24G_x\lambda_{1x}+11G_x\lambda_1^2-36G_x\lambda_5-12\lambda_{1xx}G-11\lambda_{1x}G\lambda_1+18\lambda_{5x}G=0,$ (27)

$792G_x^2\lambda_0-1056G_x^2\lambda_4+220G_x\lambda_{1y}G-330G_x\lambda_{4x}G+264G_xG\lambda_0\lambda_1-352G_xG\lambda_1\lambda_4$
$-40\lambda_{1xy}G^2-48\lambda_{1x}G^2\lambda_0+64\lambda_{1x}G^2\lambda_4-64\lambda_{1x}G^2\lambda_4-55\lambda_{4x}G^2\lambda_1+60\lambda_{5y}G^2$
$+72G^2\lambda_0\lambda_5-96G^2\lambda_4\lambda_5=0,$ (28)

$288G_x^2\lambda_{1x}+132G_x^2\lambda_1^2-432G_x^2\lambda_5-36G_x\lambda_{1xy}G-30G_x\lambda_{1x}G\lambda_1+216G_x\lambda_{5x}G+7G_xG\lambda_1^3$
$+36G_xG\lambda_1\lambda_5-648G_xG\lambda_9-36\lambda_{1xxx}G^2-42\lambda_{1xx}G^2\lambda_1-18\lambda_{1x}^2G^2-7\lambda_{1x}G^2\lambda_1^2$
$-36\lambda_{1x}G^2\lambda_5+216\lambda_{9x}G^2=0,$ (29)

$7776G_x^3\lambda_0-10368G_x^3\lambda_4+2160G_x^2\lambda_{1y}G-3240G_x^2\lambda_{4x}G+3888G_x^2G\lambda_0\lambda_1-5184G_x^2G\lambda_1\lambda_4$
$-480G_x\lambda_{1xy}G^2-864G_x\lambda_{1y}G^2\lambda_0+1152G_x\lambda_{1x}G^2\lambda_4+280G_x\lambda_{1y}G^2\lambda_1-1080G_x\lambda_{4x}G^2\lambda_1$
$+720G_x\lambda_{5y}G^2+252G_xG^2\lambda_0\lambda_1^2+1296G_xG^2\lambda_0\lambda_5-336G_xG^2\lambda_1^2\lambda_4-1728G_xG^2\lambda_4\lambda_5$
$-140\lambda_{1xy}G^3\lambda_1-120\lambda_{1xxy}G^3-216\lambda_{1xx}G^3\lambda_0+288\lambda_{1xx}G^3\lambda_4-140\lambda_{1x}\lambda_{1y}G^3$
$+120\lambda_{1x}\lambda_{4x}G^3+252\lambda_{1x}G^3\lambda_0\lambda_1+336\lambda_{1x}G^3\lambda_1\lambda_4-35\lambda_{4x}G^3\lambda_1^2-180\lambda_{4x}G^3\lambda_5$
$+720\lambda_{9y}G^3+1296G^3\lambda_0\lambda_9-1728G^3\lambda_4\lambda_9=0,$ (30)

$4G_x\lambda_{10}-\lambda_{10x}G=0,$ (31)

$\eta=(5\lambda_{10y}+8\lambda_0\lambda_{10}-9\lambda_{10}\lambda_4)/(5G^4).$ (32)

Comparing the mixed derivatives $(G_y)_{xx}=(G_{xx})_y$ and differentiating η with respect to x and y, we obtain

$-4G_x\lambda_{1y}+12G_x\lambda_{4x}+4\lambda_{1xy}G-6\lambda_{4xx}G+\lambda_{4x}G\lambda_1=0,$ (33)

$-20G_x\lambda_{10y}-32G_x\lambda_0\lambda_{10}+36G_x\lambda_{10}\lambda_4+5\lambda_{10yy}G+8\lambda_{10x}G\lambda_0-9\lambda_{10x}G\lambda_4+5\lambda_{4x}G\lambda_{10}=0,$ (34)

$\lambda_{10yy}=(-60\lambda_{10y}\lambda_0+75\lambda_{10y}\lambda_4-85\lambda_{4y}\lambda_{10}-48\lambda_0^2\lambda_{10}+68\lambda_0\lambda_{10}\lambda_4+40\lambda_{10}\lambda_2-32\lambda_{10}\lambda_4^2)/15.$ (35)

Further analysis of the compatibility depends on λ_{10}.

Case $\lambda_{10}\neq 0$

From equation (31), we find

$G_x=(\lambda_{10x}G)/4\lambda_{10}.$ (36)

Substituting G_x into equations (8), (12)-(13), (26)-(30), (33) and comparing the mixed derivatives $(G_x)_x=G_{xx}$, $(G_x)_y=(G_y)_x$, we get the conditions

$3\lambda_{10x}\lambda_0-4\lambda_{10x}\lambda_4-11\lambda_{4x}\lambda_{10}-4\lambda_1\lambda_{10}\lambda_4+4\lambda_{10}\lambda_3=0,$ (37)

λ_{4xy} $=(6\lambda_{10x}\lambda_{4y}+6\lambda_{10x}\lambda_0\lambda_4-3\lambda_{10x}\lambda_2-6\lambda_{10x}\lambda_4^2-6\lambda_{4x}\lambda_0\lambda_{10}+4\lambda_{4y}\lambda_1\lambda_{10}+4\lambda_0\lambda_1\lambda_{10}\lambda_4$
$-4\lambda_1\lambda_{10}\lambda_2-4\lambda_1\lambda_{10}\lambda_4^2+12\lambda_{10}\lambda_7)/(9\lambda_{10}),$ (38)

λ_{4xx} $=(-12\lambda_{1x}\lambda_0\lambda_{10}^2+16\lambda_{1x}\lambda_{10}^2\lambda_4+3\lambda_{10x}^2\lambda_0-4\lambda_{10x}^2\lambda_4+5\lambda_{10x}\lambda_{4x}\lambda_{10}+9\lambda_{10x}\lambda_0\lambda_1\lambda_{10}$
$-12\lambda_{10x}\lambda_1\lambda_{10}\lambda_4-20\lambda_{4x}\lambda_1\lambda_{10}^2+8\lambda_0\lambda_{10}^2\lambda_5-24\lambda_{10}^2\lambda_4\lambda_5+40\lambda_{10}^2\lambda_8)/(30\lambda_{10}^2),$ (39)

λ_{1y} $=(-9\lambda_{10x}\lambda_0+12\lambda_{10x}\lambda_4+15\lambda_{4x}\lambda_{10}-6\lambda_0\lambda_1\lambda_{10}+8\lambda_1\lambda_{10}\lambda_4)/(10\lambda_{10}),$ (40)

λ_{1xx} =$(24\lambda_{1x}\lambda_{10x} - 44\lambda_{1x}\lambda_1\lambda_{10} + 11\lambda_{10x}\lambda_1^2 - 36\lambda_{10x}\lambda_5 + 72\lambda_{5x}\lambda_{10})/(48\lambda_{10})$, (41)

λ_{5y} =$(24\lambda_{1x}\lambda_0\lambda_{10}^2 - 32\lambda_{1x}\lambda_{10}^2\lambda_4 - 3\lambda_{10x}^2\lambda_0 + 4\lambda_{10x}^2\lambda_4 - 5\lambda_{10x}\lambda_{4x}\lambda_{10} - 21\lambda_{10x}\lambda_0\lambda_1\lambda_{10}$
$+28\lambda_{10x}\lambda_1\lambda_{10}\lambda_4 + 5\lambda_{4x}\lambda_1\lambda_{10}^2 - 56\lambda_0\lambda_{10}^2\lambda_5 + 48\lambda_{10}^2\lambda_4\lambda_5 + 80\lambda_{10}^2\lambda_8)/(60\lambda_{10}^2)$, (42)

λ_{5xx} =$(16\lambda_{1x}^2\lambda_{10} - 4\lambda_{1x}\lambda_{0x}\lambda_1 + 4\lambda_{1x}\lambda_1^2\lambda_{10} - 32\lambda_{1x}\lambda_{10}\lambda_5 + 24\lambda_{10x}\lambda_{5x} - \lambda_{10x}\lambda_1^3$
$+12\lambda_{10x}\lambda_1\lambda_5 - 96\lambda_{10x}\lambda_9 - 8\lambda_{5x}\lambda_1\lambda_{10} + 128\lambda_{9x}\lambda_{10})/(32\lambda_{10})$, (43)

λ_{9y} =$(72\lambda_{1x}\lambda_{10x}\lambda_0\lambda_{10}^2 - 96\lambda_{1x}\lambda_{10}^2\lambda_4 - 80\lambda_{1x}\lambda_{4x}\lambda_{10}^3 - 12\lambda_{1x}\lambda_0\lambda_1\lambda_{10}^3 + 16\,\lambda_{1x}\lambda_1\lambda_{10}^3\lambda_4$
$-6\lambda_{10x}^3\lambda_0 + 8\lambda_{10x}^3\lambda_4 - 10\lambda_{10x}^2\lambda_{4x}\lambda_{10} - 24\lambda_{10x}^2\lambda_0\lambda_1\lambda_{10} + 32\lambda_{10x}^2\lambda_1\lambda_{10}\lambda_4$
$-20\lambda_{10x}\lambda_{4x}\lambda_1\lambda_{10}^2 + 15\lambda_{10x}\lambda_0\lambda_1^2\lambda_{10}^2 - 220\lambda_{10x}\lambda_0\lambda_{10}^2\lambda_5 - 20\lambda_{10x}\lambda_1^2\lambda_{10}^2\lambda_4$
$+300\lambda_{10x}\lambda_{10}^2\lambda_4\lambda_5 - 20\lambda_{10x}\lambda_{10}^2\lambda_8 - 40\lambda_{4x}\lambda_1^2\lambda_{10}^3 + 120\lambda_{4x}\lambda_{10}^3\lambda_5 + 264\lambda_{5x}\lambda_0\lambda_{10}^3$
$-432\lambda_{5x}\lambda_{10}^3\lambda_4 + 240\lambda_{8x}\lambda_{10}^3 + 16\lambda_0\lambda_1\lambda_{10}^3\lambda_5 - 1296\lambda_0\lambda_{10}^3\lambda_9 - 48\lambda_1\lambda_{10}^3\lambda_4\lambda_5$
$+80\lambda_1\lambda_{10}^3\lambda_8 + 1728\lambda_{10}^3\lambda_4\lambda_9)/(720\lambda_{10}^3)$, (44)

λ_{1xy} =$(-24\lambda_{1x}\lambda_0\lambda_{10}^2 + 32\lambda_{1x}\lambda_{10}^2\lambda_4 - 3\lambda_{10x}^2\lambda_0 + 4\lambda_{10x}^2\lambda_4 - 5\lambda_{10x}\lambda_{4x}\lambda_{10} + 12\lambda_{10x}\lambda_0\lambda_1\lambda_{10}$
$-16\lambda_{10x}\lambda_1\lambda_{10}\lambda_4 - 50\lambda_{4x}\lambda_1\lambda_{10}^2 + 16\lambda_0\lambda_{10}^2\lambda_5 - 48\lambda_{10}^2\lambda_4\lambda_5 + 80\lambda_{10}^2\lambda_8)/(40\lambda_{10}^2)$, (45)

λ_{10xx} =$(-8\lambda_{1x}\lambda_{10}^2 + 15\lambda_{10x}^2 + 2\lambda_{10x}\lambda_1\lambda_{10})/(12\lambda_{10})$, (46)

λ_{10xy} =$(\lambda_{10x}\lambda_{10y} - \lambda_{4x}\lambda_{10}^2)/\lambda_{10}$. (47)

Case $\lambda_{10} = 0$

From equation (26), we obtain

$$6G_x\lambda_{11} + 30\lambda_{1y}G - 45\lambda_{4x}G + G\lambda_1\lambda_{11} = 0, \quad (48)$$

where $\lambda_{11} = 18\lambda_0 - 24\lambda_4$.

In this case, the assumption $\lambda_{11} = 0$ leads to the contradiction that $G_y = 0$. Thus, we have to assume that $\lambda_{11} \neq 0$.

From equations (48), we obtain the derivative

$$G_x = G(-30\lambda_{1y} + 45\lambda_{4x} - \lambda_1\lambda_{11})/(6\lambda_{11}). \quad (49)$$

Substituting G_x into equations (8), (12)-(13), (27)-(30) and comparing the mixed derivatives $(G_x)_x = G_{xx}$, $(G_x)_y = (G_y)_x$, we obtain the conditions

$-30\lambda_{1y} - 54\lambda_{4x} - \lambda_1\lambda_{11} - 36\lambda_1\lambda_4 + 36\lambda_3 = 0$, (50)

λ_{4xy} =$(-360\lambda_{1y}\lambda_{4y} - 20\lambda_{1y}\lambda_{11}\lambda_4 + 180\lambda_{1y}\lambda_2 - 120\lambda_{1y}\lambda_4^2 + 540\lambda_{4x}\lambda_{4y} - \lambda_{4x}\lambda_{11}^2$
$+6\lambda_{4x}\lambda_{11}\lambda_4 - 270\lambda_{4x}\lambda_2 + 180\lambda_{4x}\lambda_4^2 - 6\lambda_1\lambda_{11}\lambda_2 + 36\lambda_{11}\lambda_7)/(27\lambda_{11})$, (51)

λ_{5y} =$(36\lambda_{1x}\lambda_{11}^2 - 1800\lambda_{1y}^2 + 8100\lambda_{1y}\lambda_{4x} + 510\lambda_{1y}\lambda_1\lambda_{11} - 8100\lambda_{4x}^2 - 540\lambda_{4x}\lambda_1\lambda_{11}$
$+19\lambda_1^2\lambda_{11}^2 - 84\lambda_{11}^2\lambda_5 - 720\lambda_{11}\lambda_4\lambda_5 + 2160\lambda_{11}\lambda_8)/(1620\lambda_{11})$, (52)

λ_{1xx} =$(-720\lambda_{1x}\lambda_{1y} + 1080\lambda_{1x}\lambda_{4x} - 90\lambda_{1x}\lambda_1\lambda_{11} - 330\lambda_{1y}\lambda_1^2 + 1080\lambda_{1y}\lambda_5 + 495\lambda_{4x}\lambda_1^2$
$-1620\lambda_{4x}\lambda_5 + 108\lambda_{5x}\lambda_{11} - 11\lambda_1^3\lambda_{11} + 36\lambda_1\lambda_{11}\lambda_5)/(72\lambda_{11})$, (53)

λ_{1xy} =$(-18\lambda_{1x}\lambda_{11}^2 - 900\lambda_{1y}^2 + 4050\lambda_{1y}\lambda_{4x} - 240\lambda_{1y}\lambda_1\lambda_{11} - 4050\lambda_{4x}^2 - 270\lambda_{4x}\lambda_1\lambda_{11}$
$-7\lambda_1^2\lambda_{11}^2 + 12\lambda_{11}^2\lambda_5 - 360\lambda_{11}\lambda_4\lambda_5 + 1080\lambda_{11}\lambda_8)/(540\lambda_{11})$, (54)

λ_{5xx} =$(24\lambda_{1x}^2\lambda_{11} + 120\lambda_{1x}\lambda_{1y}\lambda_1 - 180\lambda_{1x}\lambda_{4x}\lambda_1 + 10\lambda_{1x}\lambda_1^2\lambda_{11} - 48\lambda_{1x}\lambda_{11}\lambda_5 - 720\lambda_{1y}\lambda_{5x}$
$+30\lambda_{1y}\lambda_1^3 - 360\lambda_{1y}\lambda_1\lambda_5 + 2880\lambda_{1y}\lambda_9 + 1080\lambda_{4x}\lambda_{5x} - 45\lambda_{4x}\lambda_1^3 + 540\lambda_{4x}\lambda_1\lambda_5$
$-4320\lambda_{4x}\lambda_9 - 36\lambda_{5x}\lambda_1\lambda_{11} + 192\lambda_{9x}\lambda_{11} + \lambda_1^4\lambda_{11} - 12\lambda_1^2\lambda_{11}\lambda_5 + 96\lambda_1\lambda_{11}\lambda_9)/(48\lambda_{11})$, (55)

λ_{5xy} =$(810\lambda_{1x}\lambda_{1y}\lambda_{11} + 18\lambda_{1x}\lambda_1\lambda_{11}^2 + 450\lambda_{1y}^2\lambda_1 - 2025\lambda_{1y}\lambda_{4x}\lambda_1 + 75\lambda_{1y}\lambda_1^2\lambda_{11}$
$-1620\lambda_{1y}\lambda_{11}\lambda_5 + 2025\lambda_{4x}^2\lambda_1 + 135\lambda_{4x}\lambda_1^2\lambda_{11} - 162\lambda_{5x}\lambda_{11}^2 + 6480\lambda_{9y}\lambda_{11} + 2\lambda_1^3\lambda_{11}^2$
$-60\lambda_1\lambda_{11}^2\lambda_5 + 180\lambda_1\lambda_{11}\lambda_4\lambda_5 - 540\lambda_1\lambda_{11}\lambda_8 + 648\lambda_{11}^2\lambda_9)/(1620\lambda_{11})$, (56)

λ_{4xx} =$(-18\lambda_{1x}\lambda_{11}^2 + 1800\lambda_{1y}^2 - 8100\lambda_{1y}\lambda_{4x} - 150\lambda_{1y}\lambda_1\lambda_{11} + 8100\lambda_{4x}^2 - 405\lambda_{4x}\lambda_1\lambda_{11}$
$-7\lambda_1^2\lambda_{11}^2 + 12\lambda_{11}^2\lambda_5 - 360\lambda_{11}\lambda_4\lambda_5 + 1080\lambda_8\lambda_{11})/(810\lambda_{11})$, (57)

λ_{1yy} =$(1440\lambda_{1y}\lambda_{4y} - 2\lambda_{1y}\lambda_{11}^2 + 32\lambda_{1y}\lambda_{11}\lambda_4 - 720\lambda_{1y}\lambda_2 + 480\lambda_{1y}\lambda_4^2 - 2160\lambda_{4x}\lambda_{4y}$

$$+\lambda_{4x}\lambda_{11}^2 - 96\lambda_{4x}\lambda_{11}\lambda_4 + 1080\lambda_{4x}\lambda_2 - 720\lambda_{4x}\lambda_4^2 - 12\lambda_1\lambda_{11}\lambda_2 + 72\lambda_{11}\lambda_7)/(36\lambda_{11}). \quad (58)$$

All obtained results can be summarized into the following theorems:

Theorem 1. Any fourth-order ordinary differential equation

$$y^{(4)} = H(x, y, y', y'', y''') \quad (59)$$

which yields a linear equation

$$u^{(4)} + \alpha u''' + \beta u'' + \gamma u' + \eta u = \delta \quad (60)$$

via a generalized Sundman transformation

$$u(t) = F(x, y), dt = G(x, y)dx, \qquad F_y G \neq 0 \quad (61)$$

has to be of the form

$$y^{(4)} + \lambda_0(x, y)y'y''' + \lambda_1(x, y)y''' + \lambda_2(x, y)y'^2y'' + \lambda_3(x, y)y'y'' + \lambda_4(x, y)y''^2$$
$$+\lambda_5(x, y)y'' + \lambda_6(x, y)y'^4 + \lambda_7(x, y)y'^3 + \lambda_8(x, y)y'^2 + \lambda_9(x, y)y' + \lambda_{10}(x, y) = 0. \quad (62)$$

Theorem 2. Sufficient conditions for equation (62) to be linearizable via a generalized Sundman transformation (61) with $F_x = 0$ are as follows:

(a) If $\lambda_{10} \neq 0$, then the conditions are (18), (21), (24), (35), and (37)-(47).

(b) If $\lambda_{10} = 0$ and $\lambda_{11} \neq 0$, then the conditions are (18), (21), (24), and (50)-(58).

Theorem 3. Provided that the sufficient conditions in Theorem 2 are satisfied, the transformation (61) mapping equation (62) into the linear equation (60) is obtained by solving the following compatible system of equations for the functions $F(y)$ and $G(x, y)$:

(a) (16), (19), and (36);

(b) (16), (19), and (49).

EXAMPLES

Example 1. Consider the nonlinear fourth-order ordinary differential equation

$$y^{(4)} - \frac{76}{9y^2}y'^2y'' + \frac{5}{3y}y''^2 - y^{8/3}y'' + \frac{44}{9y^3}y'^4 - y^{5/3}y'^2 = 0. \quad (63)$$

Equation (63) is of the form (62) with coefficients

$$\lambda_0 = 0, \quad \lambda_1 = 0, \quad \lambda_2 = \frac{-76}{9y^2}, \quad \lambda_3 = 0, \quad \lambda_4 = \frac{5}{3y}, \quad \lambda_5 = -y^{8/3},$$
$$\lambda_6 = \frac{44}{9y^3}, \quad \lambda_7 = 0, \quad \lambda_8 = -y^{5/3}, \quad \lambda_9 = 0, \quad \lambda_{10} = 0. \quad (64)$$

It can be verified that coefficients in (64) do not satisfy the conditions of linearizability by point and contact transformations (see Ibragimov et al. (2007) and Suksern et al. (2009)). Although, we cannot apply results of Suksern and Tummakun (2014), the coefficients in (64) obey the conditions for the case (b) in Theorem 2. Thus, equation (63) is linearizable via generalized Sundman transformations.

To find the functions F and G, we have to solve the overdetermined system of partial differential equations

$$F_x = 0, \quad (65)$$
$$F_{yy} = (7F_y)/(3y), \quad (66)$$
$$G_x = 0, \quad (67)$$
$$G_y = (4G)/(3y). \quad (68)$$

Choosing the simplest solution of the system (65)-(68), i.e., $F = (3y^{10/3})\,/10$ and $G = y^{4/3}$, we obtain the linearizing generalized Sundman transformation

$$u = (3y^{10/3})\,/10,\ dt = y^{4/3}dx. \qquad (69)$$

Equations (17), (20), (22), (32), and (23) give

$$\alpha = 0, \qquad \beta = -1, \qquad \gamma = 0, \qquad \eta = 0, \qquad \delta = 0.$$

Hence, the equation (63) is mapped by the transformation (69) into the linear equation

$$u^{(4)} - u'' = 0. \qquad (70)$$

The general solution of equation (70) is

$$u = c_1e^t + c_2e^{-t} + c_3 + c_4t, \qquad (71)$$

where c_1, c_2, c_3, and c_4 are arbitrary constants. Applying the generalized Sundman transformation (69) to equation (71), we have that the general solution of equation (63) is

$$y(x) = \left[\frac{10}{3}\left(c_1e^{\phi(x)} + c_2e^{-\phi(x)} + c_3 + c_4\phi(x)\right)\right]^{\frac{3}{10}},$$

where the function $t = \phi(x)$ is a solution of the equation

$$\frac{dt}{dx} = \left[\frac{10}{3}(c_1e^t + c_2e^{-t} + c_3 + c_4t)\right]^{\frac{2}{5}}.$$

From example, if $c_1 = c_2 = c_3 = 0$ and $c_4 = \frac{3}{10}$, then we obtain the solution of equation (63) as

$$y = \left(\frac{3x}{5}\right)^{\frac{1}{2}}.$$

Example 2. Consider the nonlinear fourth-order ordinary differential equation

$$y^{(4)} + \tfrac{5}{y}y'y''' - \tfrac{3}{y^2}y'^2y'' - \tfrac{1}{y^6}y'' - \tfrac{3}{y^3}y'^4 + \tfrac{1}{y^7}y'^2 + \tfrac{1}{y^8} = 0. \qquad (72)$$

Equation (72) is of the form (62) with coefficients

$$\lambda_0 = \tfrac{5}{y}, \quad \lambda_1 = 0, \quad \lambda_2 = \tfrac{-3}{y^2}, \quad \lambda_3 = 0, \quad \lambda_4 = 0, \quad \lambda_5 = -\tfrac{1}{y^6},$$
$$\lambda_6 = -\tfrac{3}{y^3}, \quad \lambda_7 = 0, \quad \lambda_8 = \tfrac{1}{y^7}, \quad \lambda_9 = 0, \quad \lambda_{10} = \tfrac{1}{y^8}. \qquad (73)$$

It can be verified that coefficients in (73) do not satisfy the conditions of linearizability by point and contact transformations (see Ibragimov et al. (2007) and Suksernet al. (2009)). Again, we cannot apply results of Suksern and Tummakun (2014). Nevertheless, the coefficients in (73) obey the conditions for the case $\lambda_{10} \neq 0$ in Theorem 2. Thus, equation (72) is linearizable via generalized Sundman transformations.

To find the functions F and G, we have to solve the overdetermined system of partial differential equations

$$F_x = 0, \qquad (74)$$
$$F_{yy} = (-4F_y)/y, \qquad (75)$$
$$G_x = 0, \qquad (76)$$
$$G_y = (-3G)/y. \qquad (77)$$

Choosing the simplest solution of the system (74)-(77), i.e., $F = -1/(3y^3)$ and $G = 1/y^3$, we obtain the linearizing generalized Sundman transformation

$$u = -1/(3y^3),\ dt = 1/y^3dx. \qquad (78)$$

Equations (17), (20), (22), (32), and (23) give

$$\alpha = 0, \qquad \beta = -1, \qquad \gamma = 0, \qquad \eta = 0, \qquad \delta = -1.$$

Hence, the equation (72) is mapped by the transformation (78) into the linear equation

$$u^{(4)} - u'' = -1. \tag{79}$$

The general solution of equation (79) is

$$u = c_1 e^t + c_2 e^{-t} + c_3 + c_4 t + t^2/2, \tag{80}$$

where $c_1, c_2, c_3,$ and c_4 are arbitrary constants. Applying the generalized Sundman transformation (78) to equation (80), we have that the general solution of equation (72) is

$$y(x) = \left(-3\left(c_1 e^{\phi(x)} + c_2 e^{-\phi(x)} + c_3 + c_4\phi(x) + \frac{(\phi(x))^2}{2}\right)\right)^{-\frac{1}{3}}, \tag{81}$$

where the function $t = \phi(x)$ is a solution of the equation

$$\frac{dt}{dx} = -3\left(c_1 e^t + c_2 e^{-t} + c_3 + c_4 t + \frac{t^2}{2}\right).$$

AN APPLICATION TO NONLINEAR FOURTH-ORDER PARTIAL DIFFERENTIAL EQUATIONS

Nonlinear partial differential equations are encountered in various areas of model engineering. Exact solutions of partial differential equations play a significant role in the proper understanding of qualitative features of many phenomena and process in various areas of natural science. The motivation of this paper is to expand the application of the generalized Sundman transformation for finding the general solution of the most interested nonlinear fourth-order partial differential equation but with difficulty in solving:

$$u_{tt} = (\tilde{\kappa}u + \tilde{\gamma}u^2)_{xx} + \tilde{\nu}uu_{xxxx} + \tilde{\mu}u_{xxtt} + \tilde{\alpha}u_x u_{xxx} + \tilde{\beta}u_{xx}^2, \tag{82}$$

where $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\mu}, \tilde{\nu},$ and $\tilde{\kappa}$ are arbitrary constants (see Clarkson and Priestley (1999)).

We can solve this problem by the following steps:

1) reducing equation (82) to the nonlinear fourth-order ordinary differential equation by substituting the form of traveling wave solutions;

2) reducing the nonlinear fourth-order ordinary differential equation to a linear fourth-order ordinary differential equation by applying the criteria in Theorem 1, Theorem 2 and Theorem 3;

3) finding the general solution of the linear fourth-order ordinary differential equation and then substituting back to the general solution of (82).

Example 3. Consider the nonlinear fourth-order partial differential equation

$$u_{tt} = (\tilde{\kappa}u + \tilde{\gamma}u^2)_{xx} + \tilde{\nu}uu_{xxxx} + \tilde{\mu}u_{xxtt} + \tilde{\alpha}u_x u_{xxx} + \tilde{\beta}u_{xx}^2, \tag{83}$$

where $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\mu}, \tilde{\nu},$ and $\tilde{\kappa}$ are arbitrary constants. Of particular interest among solutions of equation (83) are traveling wave solutions:

$$u(x,t) = H(x - Dt),$$

where D is a constant phase velocity and the argument $x - Dt$ is a phase of the wave. Substituting the representation of a solution into equation (83), we obtain the nonlinear fourth-order ordinary differential equation

$$(\tilde{\nu}H + \tilde{\mu}D^2)H^{(4)} + \tilde{\alpha}H'H''' + \tilde{\beta}H''^2 + (2\tilde{\gamma}H + \tilde{\kappa} - D^2)H'' + 2\tilde{\gamma}H'^2 = 0. \tag{84}$$

Assume that $(\tilde{\nu}H + \tilde{\mu}D^2) \neq 0.$ Then equation (84) becomes

$$H^{(4)} + \frac{\tilde{\alpha}}{(\tilde{\nu}H + \tilde{\mu}D^2)}H'H''' + \frac{\tilde{\beta}}{(\tilde{\nu}H + \tilde{\mu}D^2)}H''^2 + \frac{(2\tilde{\gamma}H + \tilde{\kappa} - D^2)}{(\tilde{\nu}H + \tilde{\mu}D^2)}H'' + \frac{2\tilde{\gamma}}{(\tilde{\nu}H + \tilde{\mu}D^2)}H'^2 = 0. \tag{85}$$

Equation (85) is of the form (62) with coefficients

$$\lambda_0 = \frac{\tilde{\alpha}}{(\tilde{\nu}H + \tilde{\mu}D^2)}, \quad \lambda_1 = 0, \quad \lambda_2 = 0, \quad \lambda_3 = 0, \quad \lambda_4 = \frac{\tilde{\beta}}{(\tilde{\nu}H + \tilde{\mu}D^2)},$$

$$\lambda_5 = \frac{2\tilde{\gamma}H + \tilde{\kappa} - D^2}{(\tilde{\nu}H + \tilde{\mu}D^2)}, \quad \lambda_6 = 0, \quad \lambda_7 = 0, \quad \lambda_8 = \frac{2\tilde{\gamma}}{(\tilde{\nu}H + \tilde{\mu}D^2)}, \quad \lambda_9 = 0, \quad \lambda_{10} = 0. \tag{86}$$

We can check that the coefficients in (86) satisfy the conditions for the case $\lambda_{10} = 0$ and $\lambda_{11} \neq 0$ in Theorem 2 if and only if

$$\tilde{\beta} = 0, \ \tilde{\alpha} = \frac{5}{2}\tilde{\nu}, \ \tilde{\gamma} = 0, \ \tilde{\kappa} = D^2,$$

where $\tilde{\nu} \neq 0$.

Thus, the overdetermined system (86) becomes

$$\lambda_0 = \frac{5\tilde{\nu}}{2(\tilde{\nu}H + \tilde{\mu}D^2)}, \quad \lambda_1 = 0, \quad \lambda_2 = 0, \quad \lambda_3 = 0, \quad \lambda_4 = 0,$$

$$\lambda_5 = 0, \quad \lambda_6 = 0, \quad \lambda_7 = 0, \quad \lambda_8 = 0, \quad \lambda_9 = 0, \quad \lambda_{10} = 0. \tag{87}$$

with these conditions, the equation (85) becomes the nonlinear ordinary differential equation

$$H^{(4)} + \frac{5\tilde{\nu}}{2(\tilde{\nu}H + \tilde{\mu}D^2)}H'H''' = 0. \tag{88}$$

To applying the obtained results to this problem, replace x with $x - Dt$ and $y(x)$ with $H(x - Dt)$. Then equation (88) becomes

$$y^{(4)} + \frac{5\tilde{\nu}}{2(\tilde{\nu}y + \tilde{\mu}D^2)}y'y''' = 0. \tag{89}$$

To find the linearizing generalized Sundman transformation, we have to solve the overdetermined system of partial differential equations

$$F_x = 0, \tag{90}$$

$$F_{yy} = (-2\tilde{\nu}F_y)/(\tilde{\nu}y + \tilde{\mu}D^2), \tag{91}$$

$$G_x = 0, \tag{92}$$

$$G_y = (-3\tilde{\nu}G)/2(\tilde{\nu}y + \tilde{\mu}D^2). \tag{93}$$

Choosing the simplest solution of the system (90)-(93), i.e., $F = 1/(\tilde{\nu}y + \tilde{\mu}D^2)$ and $G = 1/(\tilde{\nu}y + \tilde{\mu}D^2)^{\frac{3}{2}}$, we obtain the linearizing generalized Sundman transformation

$$\tilde{u} = 1/(\tilde{\nu}y + \tilde{\mu}D^2), \ d\tilde{t} = 1/(\tilde{\nu}y + \tilde{\mu}D^2)^{\frac{3}{2}}dx. \tag{94}$$

Hence, the equation (89) is mapped by the transformation (94) into the linear equation

$$\tilde{u}^{(4)} = 0. \tag{95}$$

The general solution of equation (95) is

$$\tilde{u} = c_0 + c_1\tilde{t} + c_2\tilde{t}^2 + c_3\tilde{t}^3, \tag{96}$$

where c_0, c_1, c_2 and c_3 are arbitrary constants. Applying the generalized Sundman transformation (94) to the general solution (96), we obtain that the general solution of equation (89) is

$$y = \frac{1}{\tilde{\nu}}\left(\frac{1}{c_0 + c_1\phi(x) + c_2\phi(x)^2 + c_3\phi(x)^3} - \tilde{\mu}D^2\right), \tag{97}$$

where the function $\phi(x)$ is a solution of the equation

$$\frac{d\tilde{t}}{dx} = (c_0 + c_1\tilde{t} + c_2\tilde{t}^2 + c_3\tilde{t}^3)^{\frac{3}{2}}.$$

Hence,

$$u(x,t) = \frac{1}{\tilde{\nu}}\left(\frac{1}{c_0 + c_1\phi(x-Dt) + c_2\phi(x-Dt)^2 + c_3\phi(x-Dt)^3} - \tilde{\mu}D^2\right),$$

is the general solution of the equation

$$u_{tt} = (D^2u)_{xx} + \tilde{\nu}uu_{xxxx} + \tilde{\mu}u_{xxtt} + \frac{5}{2}\tilde{\nu}u_xu_{xxx}. \tag{98}$$

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