Γ-กึ่งกรุปที่บรรจุฐานสองด้าน

On Γ-Semigroups Containing Two-sided Bases

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บทคัดย่อ

ให้ $S = \{x, y, z, \ldots\}$ และ $\Gamma = \{\alpha, \beta, \gamma, \ldots\}$ เป็นเซตไม่ว่างใด ๆ เราเรียก S ว่า Γ-กึ่งกรุป ก็ต่อเมื่อ S สอดคล้องสองเงื่อนไขต่อไปนี้

(i) $x\alpha y \in S$ สำหรับทุก $x, y \in S$ และ $\alpha \in \Gamma$

(ii) $(x\alpha y)\beta z = x\alpha(y\beta z)$ สำหรับทุก $x, y, z \in S$ และ $\alpha, \beta \in \Gamma$

สังเกตว่าทุกกึ่งกรุปเป็น Γ-กึ่งกรุปจุดประสงค์หลักของงานนี้ อ้างถึงแนวคิดของไอดีลสองด้านที่ก่อกำเนิดโดยเซตไม่ว่าง เราแนะนำแนวคิดของฐานสองด้านของ Γ-กึ่งกรุปเราอธิบายลักษณะของเซตย่อยไม่ว่างของ Γ-กึ่งกรุปที่เป็นฐานสองด้าน พิสูจน์ว่าสำหรับสองฐานสองด้านใดๆ ของ Γ-กึ่งกรุปมีจำนวนเชิงการนับเท่ากัน และศึกษาโครงสร้างเชิงพีชคณิตของ Γ-กึ่งกรุปที่บรรจุฐานสองด้าน

ABSTRACT

Let $S = \{x, y, z, \ldots\}$ and $\Gamma = \{\alpha, \beta, \gamma, \ldots\}$ be any two non-empty sets. We call S a Γ-semigroup if S satisfies the following two conditions:

(i) $x\alpha y \in S$ for all $a, b \in S$ and $\alpha \in \Gamma$;

(ii) $(x\alpha y)\beta z = x\alpha(y\beta z)$ for all $x, y, z \in S$ and $\alpha, \beta \in \Gamma$.

It is observed that every semigroup is a Γ-semigroup. The main purpose of this paper, based on the notion of two-sided ideals generated by non-empty sets, we introduce the notion of two-sided bases of a Γ-semigroup. We characterize when a non-empty subset of a Γ-semigroup is a two-sided base, prove that for any two two-sided bases of a Γ-semigroup have the same cardinal numbers and study the algebraic structure of a Γ-semigroup containing two-sided bases.

คำสำคัญ : Γ-กึ่งกรุป ฐานสองด้าน Γ-ไอดีลสองด้าน Γ-กึ่งกรุปย่อย

Keywords: Γ-semigroup, Two-sided bases, Two-sided Γ-Ideal, Γ-subsemigroup

1. INTRODUCTION

A non-empty set S together with an associative binary operation is called a semigroup (Clifford and Preston, 1961, p.1). Based on the notion of two-sided ideals of a semigroup S generated by anon-empty set, the concept of two-sided bases of$S$ has been introduced and studied by Fabrici (Fabrici, 1975). Indeed, a non-empty subset A of S is said to be a two-sided base of S if A satisfies the following two conditions:

(i) $S = A \cup SA \cup AS \cup SAS$;

(ii) if B is a subset of A such that $S = B \cup SB \cup BS \cup SBS$, then $B = A$.

Here, for any non-empty subsets A, B of S, the set product AB is the set of all $ab \in S$ with $a \in A, b \in B$. The results obtained by Fabrici are as follows: Using the quasi-ordering defined by using principal ideals of S, the author characterized when a non-empty subset of S isa two-sided base of S, proved that for any two two-sided bases of S have the same cardinality, and studied the algebraic structure of S containing two-sided bases.

In the line of Fabricimentioned in the above paragraph, we will introduce the concept of two-sided bases of an algebraic structure which is called a Γ-semigroup. This structure was defined by Sen (Sen, 1981). Indeed, we characterize when a non-empty subset of a Γ-semigroup is a two-sided base, prove that for any two two-sided bases of a Γ-semigroup have the same cardinality and study the algebraic structure of a Γ-semigroup containing two-sided bases. It is observed that the obtained results generalize the results of Fabrici.

The rest of this section we recall definitions, notations and results used throughout the paper. Thesecan be found in (Sen and Saha, 1986; Saha, 1987; Saha, 1998). The notion of a Γ-semigroup was defined as a generalization of a semigroup by the following definition.

Definition 1.1. Let $S = \{x, y, z, \dots\}$ and $\Gamma = \{\alpha, \beta, \gamma, \dots\}$ be any two non-empty sets. Then S is said to be a Γ-semigroup if it satisfies the following two conditions:

(i) $x\alpha y \in S$ for all $x, y \in S$ and $\alpha \in \Gamma$;

(ii) $(x\alpha y)\beta z = x\alpha(y\beta z)$ for all $x, y, z \in S$ and $\alpha, \beta \in \Gamma$.

Example 1.2. Let S be a semigroup with a binary operation $\circ$. Let $\Gamma \coloneqq \{\circ\}$. Then S is a Γ-semigroup.

Example 1.3. Let S be the set of all 2×3 matrices over $\mathbb{Q}$ wher $\mathbb{Q}$ is the set of rational numbers. Let Γ be the set of all 3×2 matrices over $\mathbb{Q}$. Define $A\alpha B$ as the usual matrix multiplication of A, α, B; for all $A, B \in S$ and for all $\alpha \in \Gamma$. Then S is a Γ-semigroup. Note that S is not a semigroup under the usual matrix multiplication.

Several results on semigroups have been extended to Γ-semigroups, see for examples in (Saha, 1987; Chattopadhyay, 2005; Chinram, 2006; Sattayaporn, 2009; Hila, 2011; Hedayati and Shum, 2011).

Let S be a Γ-semigroup. For non-empty subsets A, B of S, we write $A\Gamma B$ for the set of all elements $a\alpha b$ in S where $a \in A, b \in B$ and $\alpha \in \Gamma$. That is,

$$A\Gamma B := \{a\alpha b \mid a \in A, b \in B \text{and } \alpha \in \Gamma\}.$$

For $a \in S$, we write $a\Gamma B$ for $\{a\}\Gamma B$, and similarly for $B\Gamma a$.

Definition 1.4. Let S be a Γ-semigroup and A a non-empty subset of S. Then A is called ***Γ-subsemigroup*** of Sif

$$A\Gamma A \subseteq A.$$

Definition 1.5. Let S be a Γ-semigroup and A a non-empty subset of S. Then A is called a ***left***(resp. ***right***)***Γ-ideal*** of S if

$$S\Gamma A \subseteq A \text{ (resp. } A\Gamma S \subseteq A).$$

Moreover, A is called a ***two-sidedΓ-ideal*** (or simply called an ***Γ-ideal***) of S if A is both a left and a right Γ-ideal of S.

Definition 1.6. Aproper Γ-ideal M of a Γ-semigroup S $(M \neq S)$ is said to be ***maximal*** if for anyΓ-ideal A of S,$M \subseteq A \subseteq S$ implies $M = A$ or $A = S$.

It is known (see, for instance in (Sen and Saha, 1986)) that if $\{A_i \mid i \in I\}$ is an indexed family of Γ-ideals of a Γ-semigroup S such that $\bigcap_{i\in I} A_i \neq \emptyset$, then $\bigcap_{i\in I} A_i$ is a Γ-ideal of S. Hence, if A is a non-empty subset of S, then the intersection of all Γ-ideals of S containing A, denoted by $(A)_T$, is the smallest Γ-ideal of S containing A, and $(A)_T$ is of the form

$$(A)_T = A \cup S\Gamma A \cup A\Gamma S \cup S\Gamma A\Gamma S.$$

In particular, for an element $a \in S$, we write $(\{a\})_T$ by $(a)_T$ which is called the ***principal Γ-ideal*** *of S generated by a*. Thus,

$$(a)_T = a \cup S\Gamma a \cup a\Gamma S \cup S\Gamma a\Gamma S.$$

Note that for any $b \in S$, we have

$$S\Gamma b \cup b\Gamma S \cup S\Gamma b\Gamma S$$

is a Γ-ideal of S. Finally, if A and B are twoΓ-ideals of S, then the union$A \cup B$ is a Γ-ideal of S.

2. BASES OF A Γ-SEMIGROUP

We begin this section with the definition of two-sided bases of a Γ-semigroup as follows.

Definition 2.1. Let S be a Γ-semigroup. A non-empty subset A of S is called a ***two-sided base*** of S if it satisfies the following two conditions:

(i) $S = A \cup S\Gamma A \cup A\Gamma S \cup S\Gamma A\Gamma S$;

(ii) if B is a subset of A such that $S = B \cup S\Gamma B \cup B\Gamma S \cup S\Gamma B\Gamma S$, then $B = A$.

We now provide some examples.

Example 2.2. Let $S = \{a, b, c, d\}$ and $\Gamma = \{\alpha, \beta\}$ be such that

α	a	b	c	d
a	a	b	a	a
b	b	a	b	b
c	a	b	a	a
d	a	b	a	a

β	a	b	c	d
a	b	a	b	b
b	a	b	a	a
c	b	a	b	b
d	b	a	b	b

Then S is a Γ-semigroup.It is easy to see that the two-sided base of S is $\{c, d\}$.

Example 2.3. Let $S = \{a, b, c, d\}$ and $\Gamma = \{\gamma, \delta\}$ be such that

γ	a	b	c	d
a	a	b	a	a
b	b	a	b	b
c	a	b	c	d
d	a	b	d	c

δ	a	b	c	d
a	a	b	a	a
b	b	a	b	b
c	a	b	d	c
d	a	b	c	d

Then S is a Γ-semigroup. It is easy to see that the two-sided bases of S are $\{c\}$ and $\{d\}$.

Hereafter, for any Γ-semigroup$S$, we shall use the quasi-ordering which is defined as follows.

Definition 2.4. Let S be a Γ-semigroup. We define a ***quasi-ordering*** on S by for any $a, b \in S$,

$$a \leq b \Leftrightarrow (a)_T \subseteq (b)_T.$$

We write $a < b$ if $a \leq b$ but $a \neq b$.

Lemma 2.5. Let A be a two-sided base of a Γ-semigroup S, and $a, b \in A$. If $a \in S\Gamma b \cup b\Gamma S \cup S\Gamma b\Gamma S$, then $a = b$.

Proof. Assume that $a \in S\Gamma b \cup b\Gamma S \cup S\Gamma b\Gamma S$, and suppose that $a \neq b$. Let $B = A\backslash\{a\}$. Since $a \neq b$, $b \in B$. To show that $(A)_T \subseteq (B)_T$, it suffices to show $A \subseteq (B)_T$.Let $x \in A$. If $x \neq a$, then $x \in B$, and so $x \in (B)_T$. If $x = a$, then by assumption we have $x = a \in S\Gamma b \cup b\Gamma S \cup S\Gamma b\Gamma S \subseteq (B)_T$.Thus $S = (A)_T \subseteq (B)_T \subseteq S$.This is a contradiction. Hence $a = b$.

Lemma 2.6. Let A be a two-sided base of a Γ-semigroup S. If $a, b \in A$ such that $a \neq b$ then neither $a \leq b$ nor $b \leq a$.

Proof. Assume that $a, b \in A$ such that $a \neq b$. Suppose that $a \leq b$. Then $a \in (a)_T \subseteq (b)_T$. Since $a \in (a)_T \subseteq (b)_T$ and $a \neq b$, it follows that $a \in S\Gamma b \cup b\Gamma S \cup S\Gamma b\Gamma S$. By Lemma 2.5., $a = b$. This is a contradiction. The case $b \leq a$ can be proved similarly.

3. MAIN RESULTS

In this part the algebraic structure of a Γ-semigroup containing two-sided bases will be presented.

Theorem 3.1. A non-empty subset A of a Γ-semigroup S is a two-sided base of S if and only if A satisfies the following two conditions:

(1) for any $x \in S$ there exists $a \in A$ such that $x \leq a$;

(2) for any $a, b \in A$, if $a \neq b$, then neither $a \leq b$ nor $b \leq a$.

Proof. Assume first that A is a two-sided base of S. Then $(A)_T = S$. Let $x \in S$. Then

$$x \in (A)_T = \bigcup\{\, (a)_T | \text{for all } a \in A\},$$

and so $x \in (a)_T$ for some $a \in A$. This implies $(x)_T \subseteq (a)_T$. Hence $x \leq a$. Thus (1) holds. The validity of (2) follows from Lemma 2.6..

Conversely, assume that the conditions (1) and (2) hold. We will show that A is a two-sided base of S. To show that $S = (A)_T$, let $x \in S$. By (1), there exists $a \in A$ such that $(x)_T \subseteq (a)_T$. Then

$$x \in (x)_T \subseteq (a)_T \subseteq (A)_T.$$

Thus $S \subseteq (A)_T$, and $S = (A)_T$. It remains to show that A is a minimal subset of S with the property: $S = (A)_T$. Suppose that $S = (B)_T$ for some $B \subset A$. Since $B \subset A$, there exists $a \in A \backslash B$. Since $a \in A \subseteq S = (B)_T$ and $a \notin B$, it follows that $a \in S\Gamma B \cup B\Gamma S \cup S\Gamma B\Gamma S$. There are three cases to consider:

Case1: $a \in S\Gamma B$. Then $a = s\gamma b_1$ for some $s \in S, \gamma \in \Gamma$ and $b_1 \in B$. By $a \in b_1 \cup S\Gamma b_1 \cup b_1\Gamma S \cup S\Gamma b_1\Gamma S$, it follows that $(a)_T \subseteq (b_1)_T$. Hence, $a \leq b_1$. This is a contradiction.

Case 2: $a \in B\Gamma S$. Then $a = b_2\gamma s$ for some $s \in S$, $\gamma \in \Gamma$ and $b_2 \in B$. Since $a \in b_2 \cup S\Gamma b_2 \cup b_2\Gamma S \cup S\Gamma b_2\Gamma S$, we have $(a)_T \subseteq (b_2)_T$. That is, $a \leq b_2$. This is a contradiction.

Case 3: $a \in S\Gamma B\Gamma S$. Then $a = s_1\gamma_1 b_3\gamma_2 s_2$ for some $s_1, s_2 \in S, \gamma_1, \gamma_2 \in \Gamma$ and $b_3 \in B$. By $a \in b_3 \cup S\Gamma b_3 \cup b_3\Gamma S \cup S\Gamma b_3\Gamma S$, it follows that $(a)_T \subseteq (b_3)_T$. So $a \leq b_3$. This is a contradiction.

Therefore, A is a two-sided base of S as required, and the proof is completed.

Theorem 3.2. Let A be a two-sided base of a Γ-semigroup S such that $(a)_T = (b)_T$ for some a in A and b in S. If $a \neq b$, then S contains at least two two-sided bases.

Proof. Assume that $a \neq b$. Suppose that $b \in A$. Since $a \leq b$ and $a \neq b$, it follows that

$$a \in S\Gamma b \cup b\Gamma S \cup S\Gamma b\Gamma S.$$

By Lemma 2.5., we obtain $a = b$. This is a contradiction. Thus $b \in S \backslash A$. Let

$$B := (A\backslash\{a\}) \cup \{b\}.$$

Since $b \in B$, we have $b \notin A$, and $B \nsubseteq A$. Hence $A \neq B$. We will show that B is a two-sided base of S.

To show that B satisfies (1) in Theorem 3.1., let $x \in S$. Since A is a two-sided base of S, there exists $c \in A$ such that $x \leq c$. If $c \neq a$, then $c \in B$. If $c = a$, then $x \leq a$. Since $a \leq b$, $x \leq a \leq b$. Then $x \leq b$. To show that B satisfies (2) in Theorem 3.1., let $c_1, c_2 \in B$ be such that $c_1 \neq c_2$. We will show that neither $c_1 \leq c_2$ nor $c_2 \leq c_1$. Since $c_1 \in B$ and $c_2 \in B$, we have $c_1 \in A\backslash\{a\}$ or $c_1 = b$ and $c_2 \in A\backslash\{a\}$ or $c_2 = b$.

Case 1: $c_1 \in A\backslash\{a\}$ and $c_2 \in A\backslash\{a\}$. This implies neither $c_1 \leq c_2$ nor $c_2 \leq c_1$.

Case 2: $c_1 \in A\backslash\{a\}$ and $c_2 = b$. If $c_1 \leq c_2$, then $c_1 \leq b$. Since $b \leq a$, $c_1 \leq b \leq a$. Thus$c_1 \leq a$, a contradiction. If $c_2 \leq c_1$, then $b \leq c_1$.Since $a \leq b$, $a \leq b \leq c_1$. So $a \leq c_1$, a contradiction.

Case 3: $c_2 \in A\backslash\{a\}$ and $c_1 = b$. If $c_1 \leq c_2$, then $b \leq c_2$.Since $a \leq b$, $a \leq b \leq c_2$.Hence $a \leq c_2$, a contradiction. If $c_2 \leq c_1$, then $c_2 \leq b$. Since $b \leq a$, $c_2 \leq b \leq a$. Thus $c_2 \leq a$, a contradiction.

Case 4: $c_1 = b$ and $c_2 = b$. This is impossible.

Therefore, B is a two-sided base of S.

Corollary 3.3. Let A be a two-sided base of a Γ-semigroup S, and let $a \in A$. If $(x)_T = (a)_T$ for some $x \in S$, $x \neq a$, then x belongs to some two-sided base of S, which is different from A.

Theorem 3.4. Let A and B be any two-sided bases of a Γ-semigroup S. Then A and B have the same cardinality.

Proof. Let $a \in A$. Since B is a two-sided base of S, by Theorem 3.1.(1) there exists an element $b \in B$ such that $a \leq b$. Since A is a two-sided base of S, by Theorem 3.1.(1) there exists $a^* \in A$ such that $b \leq a^*$. So $a \leq b \leq a^*$, i.e., $a \leq a^*$. This implies that $a = a^*$. Hence $(a)_T = (b)_T$. Define a mapping

$$\varphi : A \longrightarrow B \text{ by} \varphi(a) = b$$

for all $a \in A$. We will show that φ is one-one. Let $a_1, a_2 \in A$ be such that $\varphi(a_1) = \varphi(a_2)$. Since $\varphi(a_1) = \varphi(a_2)$, $\varphi(a_1) = \varphi(a_2) = b$ for some $b \in B$. So $(a_2)_T = (a_1)_T = (b)_T$. Since $(a_2)_T = (a_1)_T$, $a_1 \leq a_2$ and $a_2 \leq a_1$. This implies $a_1 = a_2$. Therefore, φ is one-one. We will show that φ is onto. Let $b \in B$. Since A is a two-sided base of S, by Theorem 3.1.(1) there exists an element $a \in A$ such that $b \leq a$. Since B is a two-sided base of S, by Theorem 3.1.(1) there exists an element $b^* \in B$ such that $a \leq b^*$. So $b \leq a \leq b^*$, i.e., $b \leq b^*$. This implies $b = b^*$. Hence $(a)_T = (b)_T$. Thus $\varphi(a) = b$. Therefore, φ is onto. This completes the proof.

Remark 3.5. It is observed that a two-sided base A of a Γ-semigroup S is a two-sided Γ-ideal of S if and only if $A = S$.

Theorem 3.6. A two-sided base A of a Γ-semigrous S is a Γ-subsemigroup if and only if $A = \{a\}$ with $a\gamma a = a$ for all $\gamma \in \Gamma$.

Proof. Assume that A of a Γ-subsemigroup S. Let $a, b \in A$ and $\gamma \in \Gamma$. Since A is a Γ-subsemigroup of S, $a\gamma b \in A$. Setting $a\gamma b = c$; thus

$$c \in S\Gamma b \subseteq S\Gamma b \cup b\Gamma S \cup S\Gamma b\Gamma S.$$

By Lemma 2.5., $c = b$. So $a\gamma b = b$. Similarly, $c \in a\Gamma S \subseteq S\Gamma a \cup a\Gamma S \cup S\Gamma a\Gamma S$. By Lemma 2.5., $c = a$.

So $a\gamma b = a$. We have $a = b$. Therefore, $A = \{a\}$ with $a\gamma a = a$.The converse statement is clear.

Notation. The union of all two-sided bases of a Γ-semigroup S is denoted by $\mathcal{A}$.

Theorem 3.7. Let S be a Γ-semigroup. Then $S\backslash\mathcal{A}$ is either empty set or a Γ-ideal of S.

Proof. Assume that $S\backslash\mathcal{A} \neq \emptyset$. We will show that $S\backslash\mathcal{A}$ is a Γ-ideal of S. Let $a \in S\backslash\mathcal{A}, x \in S$ and $\gamma \in \Gamma$. To show that $x\gamma a \in S\backslash\mathcal{A}$ and $a\gamma x \in S\backslash\mathcal{A}$, we suppose that $x\gamma a \notin S\backslash\mathcal{A}$. Then $x\gamma a \in \mathcal{A}$. Hence $x\gamma a \in A$ for some a two-sided base A of S. We set $x\gamma a = b$. Then $b \in S\Gamma a$. By

$$b \in S\Gamma a \subseteq a \cup S\Gamma a \cup a\Gamma S \cup S\Gamma a\Gamma S,$$

it follows that $(b)_T \subseteq (a)_T$. Next, we will show that $(b)_T \subset (a)_T$. Suppose that $(b)_T = (a)_T$. Since $a \in S\backslash\mathcal{A}$ and $b \in A$, $a \neq b$. Since $(b)_T = (a)_T$, $a \neq b$ and Corollary 3.3., we conclude that $a \in \mathcal{A}$. This is a contradiction. Thus $(b)_T \subset (a)_T$, i.e., $a < b$. Since A is a two-sided base of S and $a \in S\backslash\mathcal{A}$, by Theorem 3.1.(1) there exists $b_1 \in A$ such that $a \leq b_1$. Since $b < a \leq b_1$, $b \leq b_1$. This is a contradiction. Thus $x\gamma a \in S\backslash\mathcal{A}$. Similarly, $a\gamma x \in S\backslash\mathcal{A}$. Therefore, $S\backslash\mathcal{A}$ is a Γ-ideal of S.

Notation. Let $\mathcal{M}^*$ be a proper Γ-ideal of a Γ-semigroup S containing every proper Γ-ideal of S.

Theorem 3.8. Let S be a Γ-semigroup and $\emptyset \neq \mathcal{A} \subset S$. The following statements are equivalent:

(1) $S\backslash\mathcal{A}$ is a maximal proper two-sided Γ-ideal of S;

(2) for every element $a \in \mathcal{A}, \mathcal{A} \subseteq (a)_T$;

(3) $S\backslash\mathcal{A} = \mathcal{M}^*$;

(4) every two-sided base of S is a one-element base.

Proof. (1)⇔(2). Assume that $S\backslash\mathcal{A}$ is a maximal proper Γ-ideal of S. Let $a \in \mathcal{A}$. Suppose that $\mathcal{A} \nsubseteq (a)_T$. Since $\mathcal{A} \nsubseteq (a)_T$, there exists $x \in \mathcal{A}$ such that $x \notin (a)_T$. So $x \notin S\backslash\mathcal{A}$. Since $x \notin (a)_T$, $x \notin S\backslash\mathcal{A}$ and $x \in S$, we have$(S\backslash\mathcal{A}) \cup (a)_T \subset S$. Thus $(S\backslash\mathcal{A}) \cup (a)_T$ is a proper Γ-ideal of S. Hence $S\backslash\mathcal{A} \subset (S\backslash\mathcal{A}) \cup (a)_T$. This contradicts to the maximality of $S\backslash\mathcal{A}$.

Conversely, assume that for every element $a \in \mathcal{A}, \mathcal{A} \subseteq (a)_T$. We will show that $S\backslash\mathcal{A}$ is a maximal proper Γ-ideal of S. Since $a \in \mathcal{A}$, $a \notin S\backslash\mathcal{A}$. So $S\backslash\mathcal{A} \subset S$. Since $\mathcal{A} \subset S, S\backslash\mathcal{A} \neq \emptyset$. By Theorem 3.7., $S\backslash\mathcal{A}$ is a proper Γ-ideal of S. Suppose that M is aproper Γ-ideal of S such that $S\backslash\mathcal{A} \subset M \subset S$. Since $S\backslash\mathcal{A} \subset M$, there exists $x \in M$ such that $x \notin S\backslash\mathcal{A}$, i.e., $x \in \mathcal{A}$. Then $x \in M \cap \mathcal{A}$. So $M \cap \mathcal{A} \neq \emptyset$. Let $c \in M \cap \mathcal{A}$. Then $c \in M$ and $c \in \mathcal{A}$. Since $c \in M$,

$$S\Gamma c \subseteq S\Gamma M \subseteq M,\ c\Gamma S \subseteq M\Gamma S \subseteq M \text{ and } S\Gamma c\Gamma S \subseteq S\Gamma M\Gamma S \subseteq M.$$

Then $(c)_T = c \cup S\Gamma c \cup c\Gamma S \cup S\Gamma c\Gamma S \subseteq M$. Since $c \in \mathcal{A}$, by assumption we have $\mathcal{A} \subseteq (c)_T$. Hence

$$S = (S\backslash\mathcal{A}) \cup \mathcal{A} \subseteq (S\backslash\mathcal{A}) \cup (c)_T \subseteq M \subset S.$$

Thus $M = S$. This is a contradiction. Therefore, $S\backslash\mathcal{A}$ is a maximal Γ-ideal of S.

(3)⇔(4). Assume that $S\backslash\mathcal{A} = \mathcal{M}^*$. Since $S\backslash\mathcal{A} = \mathcal{M}^*$, $S\backslash\mathcal{A}$ is a maximal proper Γ-ideal of S. By (1)⇔(2), for every $a \in \mathcal{A}, \mathcal{A} \subseteq (a)_T$. First, we will show that for every $a \in \mathcal{A}, S\backslash\mathcal{A} \subseteq (a)_T$. Suppose that $S\backslash\mathcal{A} \nsubseteq (a)_T$ for some $a \in \mathcal{A}$. Then $(a)_T \neq S$. Hence $(a)_T$ is a proper Γ-ideal of S. Thus $(a)_T \subseteq \mathcal{M}^* = S\backslash\mathcal{A}$. Then $(a)_T \subseteq S\backslash\mathcal{A}$. Since $a \in (a)_T$, $a \in S\backslash\mathcal{A}$, i.e., $a \notin \mathcal{A}$. This is a contradiction. Thus $S\backslash\mathcal{A} \subseteq (a)_T$ for every $a \in \mathcal{A}$. Since $S\backslash\mathcal{A} \subseteq (a)_T$ and $\mathcal{A} \subseteq (a)_T$ for every $a \in \mathcal{A}$, it follows that

$$S = (S\backslash\mathcal{A}) \cup \mathcal{A} \subseteq (a)_T \cup (a)_T = (a)_T \subseteq S.$$

So $S = (a)_T$ for every $a \in \mathcal{A}$. Therefore, $\{a\}$ is a two-sided base of S. Let A be a two-sided base of S.

We will show that $a = b$ for all $a, b \in A$. Suppose that there exist $a, b \in A$ such that $a \neq b$. Since A is a two-sided base of S, $A \subseteq \mathcal{A}$. That is, $a \in \mathcal{A}$. So $S = (a)_T$. Since $b \in S = (a)_T$ and $b \neq a$, $b \in S\Gamma a \cup a\Gamma S \cup S\Gamma a\Gamma S$. By Lemma 2.5., $a = b$. This is a contradiction. Therefore, every two-sided base of S is a oneelement base.

Conversely, assume that every two-sided base of S is a oneelement base. Then $S = (a)_T$ for all $a \in \mathcal{A}$. We will show that $S \backslash \mathcal{A} = \mathcal{M}^*$. The statement that $S \backslash \mathcal{A}$ is a maximal proper Γ-ideal of S follows from the proof (1)⇔(2). Let M be a Γ-ideal of S such that M is not contained in $S \backslash \mathcal{A}$. Then $\mathcal{A} \cap M \neq \emptyset$. Let $a \in \mathcal{A} \cap M$. Then $a \in \mathcal{A}$ and $a \in M$. It follows that

$$S\Gamma a \subseteq S\Gamma M \subseteq M,\ a\Gamma S \subseteq M\Gamma S \subseteq M \text{ and } S\Gamma a\Gamma S \subseteq S\Gamma M\Gamma S \subseteq M.$$

Then $S = (a)_T \subseteq M \subseteq S$. Thus $M = S$.

(1)⇔(3). Assume that $S \backslash \mathcal{A}$ is a maximal proper Γ-ideal of S. We will show that $S \backslash \mathcal{A} = \mathcal{M}^*$. Since $S \backslash \mathcal{A}$ is a proper Γ-ideal of S, $S \backslash \mathcal{A} \subseteq \mathcal{M}^* \subset S$. By assumption, $S \backslash \mathcal{A} = \mathcal{M}^*$or $S = \mathcal{M}^*$. Hence $S \backslash \mathcal{A} = \mathcal{M}^*$. The converse statement is obvious.

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