ไอดีลใหญ่สุดเฉพาะกลุ่มพัวส์ซองและมอดูลพัวส์ซองเชิงเดียวที่มีมิติจำกัดของพีชคณิตพัวส์ซองบางรูป

Poisson Maximal Ideals and the Finite-dimensional Simple Poisson Modules of a Certain Poisson Algebra

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บทคัดย่อ

เราศึกษาพีชคณิตพัวส์ซอง $\mathrm{A} = \mathbb{C}[\mathrm{x}, \mathrm{y}, \mathrm{z}]$ พร้อมด้วย $\{-,-\}_A$ บางรูป และแสดงว่า พีชคณิตพัวส์ซอง A นี้มี ไอดีลใหญ่สุดเฉพาะกลุ่มพัวส์ซองเพียง 2 ตัวกล่าวคือ J_1 และ J_2 เราทราบว่า $\mathrm{J}/_{\mathrm{J}^2}$ มีโครงสร้างของพีชคณิตลี เมื่อ J คือ ไอดีลใหญ่สุดเฉพาะกลุ่มพัวส์ซองของ A แนวคิดนี้นำไปสู่การหามอดูลพัวส์ซองเชิงเดียวที่มีมิติจำกัดของพีชคณิตพัวส์ซอง A ผลที่ได้สำหรับ J_1 คือ ทุกๆ มอดูลพัวส์ซองเชิงเดียวของพีชคณิตพัวส์ซอง A ที่ถูกกำจัดด้วย J_1 จะมีเพียง 1 มิติ และผลที่ได้สำหรับ J_2 คือ สำหรับแต่ละ $d \geq 1$, มีมอดูลพัวส์ซองเชิงเดียวของพีชคณิตพัวส์ซอง A ที่ถูกกำจัดด้วย J_2 อยู่เพียง 1 ตัวที่มี d มิติ

ABSTRACT

We examine a Poisson algebra of the form $\mathrm{A} = \mathbb{C}[\mathrm{x}, \mathrm{y}, \mathrm{z}]$ with a certain bracket $\{-,-\}_A$, and show that there are only two Poisson maximal ideals of A, namely J_1 and J_2. It is known that $\mathrm{J}/_{\mathrm{J}^2}$ has a natural Lie algebra structure where J is a Poisson maximal ideal of A . This idea leads to determine the finite-dimensional Simple Poisson A-modules. For the maximal ideals J_1 of A, we show that every finite-dimensional simple Poisson module over A annihilated by J_1 is one-dimensional. For the maximal ideals J_2 of A, we show that for $d \geq 1$, there is a unique d-dimensional simple Poisson module over A annihilated by J_2.

คำสำคัญ: พีชคณิตพัวส์ซอง ไอดีลพัวส์ซอง ไอดีลใหญ่ที่สุดเฉพาะกลุ่มพัวส์ซอง พีชคณิตอนุพัทธ์ มอดูลพัวส์ซอง

Keywords: Poisson algebras, Poisson ideal, Poisson maximal ideal, derived algebra, Poisson modules

1. INTRODUCTION

A Poisson algebra is the associative algebra having a Lie algebra structure together with the Leibniz law. Sasom (2006) considered Poisson $\mathbf{A}$-modules in the aspect of Farkas (2000) and Oh (1999) by using direct method to classify the finite-dimensional simple Poisson modules for the Poisson algebra $\mathrm{A} = \mathbb{C}[\mathrm{x}, \mathrm{y}, \mathrm{z}]$ with Poisson bracket:

$$\{x, y\} = yx + z,\ \{y, z\} = zy + x \text{ and } \{z, x\} = xz + y,$$

for all $\mathrm{x}, \mathrm{y}, \mathrm{z} \in \mathrm{A}$.

The aim of this research is to classify the finite-dimensional simple Poisson modules for the Poisson algebra with the certain Poisson bracket, namely, the Poisson algebra $\mathrm{A} = \mathbb{C}[\mathrm{x}, \mathrm{y}, \mathrm{z}]$ with Poisson bracket:

$$\{x, y\} = yx + x + y + z,\ \{y, z\} = zy + x + y + z \text{ and } \{z, x\} = xz + x + y + z,$$

for all $\mathrm{x}, \mathrm{y}, \mathrm{z} \in \mathrm{A}$.

The more complicated Poisson bracket is the obstacle to classify the finite-dimensional simple Poisson modules for A if we use the direct method appeared in Sasom (2006). We find the new method in order to apply it to obtain this research results. This method is presented by Jordon (2010). We can see that if J is a Poisson maximal ideal of A, then ${}^{\mathrm{J}}/_{\mathrm{J}^2}$ has a natural Lie algebra structure. The valuable result is shown in Jordan (2010) stating that there is a bijection, which preserves dimension, between the isomorphism classes of finite-dimensional simple Poisson A-modules and pairs $(J, \widehat{M})$ where J is a Poisson maximal ideal of A and $\widehat{M}$ is an isomorphism class of finite-dimensional modules over the Lie algebra ${}^{\mathrm{J}}/_{\mathrm{J}^2}$. In this bijection, the simple Poisson modules in a class corresponding to the pairs $(J, \widehat{M})$ are annihilated by J. Hence if the Poisson maximal ideals J of A can be identified and the representation theory of the Lie algebra $\mathfrak{g}(\mathrm{J})$ is known, then the finite-dimensional simple Poisson A-modules can be seen.

Note that a $\mathbb{C}$-algebra T is called t-homogeneous if, for each positive integer d, there are, up to isomorphism, precisely t simple (left) T-modules of dimension d and that a Poisson algebra is t-homogeneous if it has the analogous Poisson property.

2. PRELIMINARIES

In this topic, it contains some of the materials that will be used throughout this work. The main topics are Lie algebra, Derived algebra, Low-dimensional Lie algebra, Poisson algebra and Poisson module and Jordan's result shown in Jordan (2010).

We now give the definition of Lie algebra which is the foundation of the main topic, a Poisson Algebra.

Definition 2.1 Let F be a field. A **Lie algebra** over F is an F-vector space L, together with a bilinear map, the Lie bracket $\mathrm{L} \times \mathrm{L} \longrightarrow \mathrm{L}$ given by $(\mathrm{x}, \mathrm{y}) \mapsto [\mathrm{x}, \mathrm{y}]$ satisfying the following properties:

1) $[x, x] = 0$ for all $x \in L$,
2) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in L$.

Example 2.2 The well-known vector space of all $\mathrm{n} \times \mathrm{n}$ matrices over a field F, denoted by $\mathbf{gl}(\mathbf{n}, \mathbf{F})$, has the Lie bracket defined by $[\mathrm{x}, \mathrm{y}] = \mathrm{xy} - \mathrm{yx}$, where xy is the usual product of the matrices x and y. So $(\mathrm{gl}(\mathrm{n}, \mathrm{F}), [-,-])$

is a Lie algebra. It is called ***general linear algebra***. Another well-known algebra is the subspace of $\mathrm{gl}(\mathrm{n},\mathrm{F})$ consisting of all $\mathrm{n}\times\mathrm{n}$ matrices of trace 0 named as the **special linear algebra** and denoted by $\mathrm{sl}(\mathrm{n},\mathrm{F})$.

The derived algebra is one of the tools for investigating the results. Let I and J be ideals of a Lie algebra L. The product of ideals $[\mathrm{I},\mathrm{J}]$ is the span of the commutators of elements of I and J, that is $[\mathrm{I},\mathrm{J}] \coloneqq$ Span $\{[\mathrm{x},\mathrm{y}]: \mathrm{x}\in\mathrm{I}, \mathrm{y}\in\mathrm{J}\}$.

Definition 2.3 If $\mathrm{I}=\mathrm{J}=\mathrm{L}$, then the ideal $[\mathrm{L},\mathrm{L}]$ is called the **derived algebra** of L.

Example 2.4 $[\mathrm{gl}(\mathrm{n},\mathrm{F}),\mathrm{gl}(\mathrm{n},\mathrm{F})]=\mathrm{sl}(\mathrm{n},\mathrm{F})$ and $[\mathrm{sl}(\mathrm{n},\mathrm{F}),\mathrm{sl}(\mathrm{n},\mathrm{F})]=\mathrm{sl}(\mathrm{n},\mathrm{F})$.

Low-dimensional Lie algebra

The basic way to find how many non-isomorphic Lie algebras there are in order to classify them is to understand the low-dimensions. The reason to work on the low-dimensional Lie algebras is that they often appear as subalgebras of the larger Lie algebras. We shall look at the Lie algebras of dimension 1, 2 and 3. All the results provided in this topic is from Erdmann and Wildon (2006).

We can see easily that every 1 dimensional Lie algebras is abelian. For any field F, up to isomorphism, there is a unique 2-dimensional non-abelian Lie algebra over F. This Lie algebra has a basis $\{\mathrm{x},\mathrm{y}\}$ such that its Lie bracket is described by $[\mathrm{x},\mathrm{y}]=\mathrm{x}$. The center of this Lie algebras is 0. For 3-dimensional Lie algebras, we focus here for 2 cases. For other cases, one can study more in Erdmann and Wildon (2006).

Firstly, the 3 dimensional Lie algebra L which its derived algebra $[\mathrm{L},\mathrm{L}]$ has dimension 1 and $[\mathrm{L},\mathrm{L}]\subseteq \mathrm{Z}(\mathrm{L})$, the center of L, appears uniquely and it has a basis $\{\mathrm{f},\mathrm{g},\mathrm{h}\}$ where $[\mathrm{f},\mathrm{g}]=\mathrm{h}\in\mathrm{Z}(\mathrm{L})$. This Lie algebra is known as the **Heisenberg algebras**. For another one, suppose that L is a complex Lie algebra of dimension 3 such that $[\mathrm{L},\mathrm{L}]=\mathrm{L}$. Up to isomorphism, $\mathrm{sl}(2,\mathbb{C})$ is the unique 3-dimensional Lie algebras L with $[\mathrm{L},\mathrm{L}]=\mathrm{L}$.

Definition 2.5 Let A be a finitely generated commutative algebra over $\mathbb{C}$. A ***Poisson bracket*** on A is a Lie algebra bracket $\{-,-\}$ satisfying the Leibniz rule: $\{ab,c\}=a\{b,c\}+\{a,c\}b$ for all $a,b,c\in A$. The pair $(A,\{-,-\})$ is called a **Poisson algebra.**

Definition 2.6 A subalgebra B of A is a ***Poisson subalgebra*** of A if $\{b,c\}\in B$ for all $b,c\in B$. An ideal I of a Poisson subalgebra A is a **Poisson ideal** if $\{i,a\}\in I$ for all $i\in I$ and all $a\in A$.

If I is a Poisson ideal of A then A/I is a Poisson algebra in the natural way:

$$\{a+I,b+I\}=\{a,b\}+I$$

Definition 2.7 An ideal I of a Poisson algebra A is said to be a ***Poisson maximal ideal*** if I is a maximal ideal of A and also a Poisson ideal.

In the literature, there is a similar definition to the Poisson maximal ideal. We will mention it here. By ***maximal Poisson ideal***, we shall mean a Poisson ideal I of A such that if J is a Poisson ideal and $I\subseteq J$ then $J=A$. For example, let $A=\mathbb{C}[x,y]$ which is a Poisson algebra with the Poisson bracket $\{x,y\}=1$. Then 0 is a maximal Poisson ideal but is not a Poisson maximal ideal.

Next, it is the important Theorem concerning the maximal ideal of a polynomial ring.

Theorem 2.8 (Hilbert's Nullstellensatz Theorem) Let $\mathrm{F}[\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_3,\ldots,\mathrm{x}_\mathrm{n}]$ be that polynomial ring over a field F in the inderminates $\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_3,\ldots,\mathrm{x}_\mathrm{n}$. The ideal M is a maximal ideal if and only if there exist $\mathrm{a}_1,\mathrm{a},\mathrm{a}_3,\ldots,\mathrm{a}_\mathrm{n}$ such that $\mathrm{M}=\langle \mathrm{x}_1-\mathrm{a}_1,\mathrm{x}_2-\mathrm{a}_2,\mathrm{x}_3-\mathrm{a}_3,\ldots,\mathrm{x}_\mathrm{n}-\mathrm{a}_\mathrm{n}\rangle$.

Proof. See Sharp (2000) Theorem 14.6.

Next, we will give the definition of Poisson module. There is more than one definition of Poisson module in the literature. We shall use the one introduced in Farkas (2000) and Oh (1999).

Definition 2.9 A module M over a Lie algebra $\mathcal{g}$ (M is also called ***$\mathcal{g}$-module***) is a $\mathbb{C}$-vector space together with a bilinear form $[-,-]_M : \mathcal{g} \times M \to M$ such that

$$[[a,b],m]_M = [a,[b,m]]_M - [b,[a,m]]_M$$

for all $a, b \in \mathcal{g}$ and $m \in M$. A subspace V of M is a ***submodule*** of M if $[a,v]_M \in V$ for all $a \in \mathcal{g}$ and all $v \in V$. M is ***simple*** if $M \neq 0$ and its only submodules are only 0 and M.

Definition 2.10 Let A be a commutative Poisson algebra with Poisson bracket $\{-,-\}$. An A-module M is a ***Poisson A-module*** if there is a bilinear form $\{-,-\}_M : A \times M \to M$ such that the following axioms hold for all $a, a' \in A$ and all $m \in M$:

(i) $\{a,a'\}_M = \{a,a'\}\, m + a'\{a,m\}_M$;

(ii) $\{aa',m\}_M = a\,\{a',m\}_M + a'\{a,m\}_M$;

(iii) $\{\{a,a'\},m\}_M = \{a,\{a',m\}_M\}_M - \{a',\{a,m\}_M\}_M$

A submodule N of a Poisson module M is called a **Poisson submodule** if $\{a,n\}_M \in N$ for all $a \in A$ and $n \in N$.

Definition 2.11 Let N be a left module over a ring R. Give any subset $X \subseteq N$, the **annihilator** of X is the set $ann_R(X) = \{r \in R : rx = 0 \text{ for all } x \in X\}$, which is a left ideal of R.

Let M be a Poisson module over a Poisson algebra A and let $S \subseteq M$. In the module sense, we denote the annihilator of S in A by $ann_A(S)$. And we denote

$$Pann_A(S) = \{a \in A : \{a,m\}_M = 0 \text{ for all } m \in S\}.$$

Lemma 2.12 Let A be a Poisson algebra and M be a Poisson A-module.

1. The annihilator $ann_A(M)$ is a Poisson ideal of A.
2. if M is a finite-dimensional simple Poisson module then $ann_A(M)$ is a Poisson maximal ideal of A.
3. $\mathbb{C} + J^2 \subseteq Pann_A M$.

Proof. See Jordan (2010) Lemma 1.

Let $(A,\{-,-\})$ be Poisson algebra and let I and J be Poisson ideals of A. Then IJ is a Poisson ideal of A. Of course I and J are Lie subalgebra of A under $\{-,-\}$. If $I \subseteq J$, then I is a Lie ideal of J and J/I is a Lie algebra. In particular, J/J^2 is always a Lie algebra.

Studying Poisson modules, one natural way to find Poisson modules is, for I and J are Poisson ideals of A with $I \subseteq J$, the factor J/I is a Poisson A-module with, for $a \in A$ and $j \in J$,

$$\{a, j+I\}_{J/I} = \{a,j\}_J + I.$$

We can check that $\{-,-\}_{J/I}$ is well-defined, and all the axioms for a Poisson module are hold. By above argument, J/I is also a Lie algebra. Every Poisson subalgebra of J/I is a Lie ideal, so if J/I is simple as a Lie algebra, then it is simple as a Poisson module. If A is affine and J is a Poisson maximal ideal, so that $A = J + \mathbb{C}$, then the converse is also true because every Lie ideal of J/I is then a Poisson A-submodule. If I and J are Poisson ideals of a Poisson algebra A, then I/IJ and J/IJ are Poisson modules (by affine Poisson algebra, we mean a Poisson algebra that is finitely generated as a $\mathbb{C}$-algebra).

The following is the main result by Jordan (2010). We use this result to tackle the later research problems. Jordan proves the result giving a method to determine the finite-dimensional simple Poisson modules over any affine Poisson algebra as the following Theorem.

Theorem 2.13 Let A be an affine generated Poisson algebra.

(i) Let M be a finite-dimensional simple Poisson A -module and let $J = ann_A(M)$. There is a simple module M^* for the Lie algebra $\mathcal{g}(J)$ such that $M^* = M$, as $\mathbb{C}$-vector space, and $[j + J^2, m]_{M^*} = \{j, m\}_M$ for all $j \in J$ and $m \in M$.

(ii) Let J be a Poisson maximal ideal A of and let N be a finite-dimensional simple $\mathcal{g}(J)$-module. There exist a simple Poisson A-module N' and a Lie homomorphism $f: A \to \mathcal{g}(J)$ such that $N' =\ {}^fN$ \$ as a Lie module over A and $J = ann_A(N')$.

(iii) For all finite-dimensional simple Poisson modules M, $M^{*\prime} = M$. For all Poisson maximal ideals J of A and all finite-dimensional simple $\mathcal{g}(J)$-modules N, $N' = N$.

(iv) The procedure in (i) and (ii) establish a bijection Γ from the set of isomorphism classes of finite-dimensional simple Poisson module over A to the set of pairs $(J, \widehat{N})$, where J is a Poisson maximal ideal of A and N is a finite-dimensional simple $\mathcal{g}(J)$-module, given by $\Gamma(\widehat{M}) = (ann_A(M), \widehat{M^*})$.

Proof. See Jordan (2010) Theorem 1.

3. MAIN RESULTS

In this section, we classify the finite-dimensional simple Poisson modules over the certain Poisson algebra A. Let we start by giving S as the $\mathbb{C}$-algebra generated by $\mathrm{x, y, z, q}$ and q^{-1} subject to the relations

$$\mathrm{xy - qyx = (q-1)(x+y+z)},$$
$$\mathrm{yz - qzy = (q-1)(x+y+z)},$$
$$\mathrm{zx - qxz = (q-1)(x+y+z)},$$
$$\mathrm{xq = qx,\ yq = qy,\ zq = qz,} \text{ and } \mathrm{qq^{-1} = 1 = q^{-1}q}.$$

Then q is a central element of S. Let $\mathrm{A} = {}^{\mathrm{S}}/_{(\mathrm{q}-1)\mathrm{S}} \simeq \mathbb{C}[\mathrm{x, y, z}]$, which is a commutative algebra. The induced Poisson bracket on A is such that

$$\{\mathrm{x, y}\} = \frac{1}{\mathrm{q}-1}[\mathrm{x, y}] = \frac{1}{\mathrm{q}-1}(\mathrm{xy - yx}) = \frac{1}{\mathrm{q}-1}\big(\mathrm{qyx - yx + (q-1)(x+y+z)}\big) = \mathrm{yx + x + y + z}.$$

Similarly, we obtain $\{\mathrm{y, z}\} = \mathrm{zy + x + y + z}$ and $\{\mathrm{z, x}\} = \mathrm{xz + x + y + z}$. Hence, these are the Poisson brackets on A :

$$\begin{aligned} \{\mathrm{x, y}\} &= \mathrm{yx + x + y + z}, \\ \{\mathrm{y, z}\} &= \mathrm{zy + x + y + z}, \\ \{\mathrm{z, x}\} &= \mathrm{xz + x + y + z}. \end{aligned} \tag{3.1}$$

Next, we examine the Poisson maximal ideals of A with the Poisson brackets (3.1).

Lemma 3.1 Let A be the Poisson algebra with the Poisson brackets (3.1). There are only two Poisson maximal ideals of A. More precisely, they are:

$$\mathrm{J_1 = xA + yA + zA} \text{ and } \mathrm{J_2 = (x+3)A + (y+3)A + (z+3)A}.$$

Proof. By Theorem 2.8, we have $J = (x-a)A + (y-b)A + (z-c)A$ as a Poisson maximal ideal of A, for suitable $a, b, c \in \mathbb{C}$. Since J is a Poisson ideal, $\{x, J\} \subseteq J, \{y, J\} \subseteq J$ and $\{z, J\} \subseteq J$. Observe that

$$J \supseteq \{x, y-b\} = \{x,y\} - \{x,b\} = \{x,y\} = yx + x + y + z,$$
$$J \supseteq \{y, z-c\} = \{y,z\} - \{y,c\} = \{y,z\} = zy + x + y + z,$$
$$J \supseteq \{z, x-a\} = \{z,x\} - \{z,a\} = \{z,x\} = xz + x + y + z.$$

By the above three equations, we obtain

$$ab + a + b + c = 0, \quad (3.2)$$

$$bc + a + b + c = 0, \quad (3.3)$$

$$ac + a + b + c = 0, \quad (3.4)$$

It induces that $ab - bc = 0$ which implies that $b = 0$ or $a = c$, and

$ab - ac = 0$ which implies that $a = 0$ or $b = c$, and

$bc - ac = 0$ which implies that $c = 0$ or $a = b$.

There are two cases to be considered.

Case 1. $b = 0$. The equation (3.2) gives $a = -c$. Then substitute this in the equation (3.4), we have $c = 0$, which gives $a = 0$.

Case 2. $b \neq 0$ and $a = c$. The equation (4.1) gives $ab + 2a + b = 0$, and the equation (3.4) gives $a^2 + 2a + b = 0$. These arguments give $0 = a^2 - ab = a(a-b)$. Thus $a = 0$ or $a = b$. If $c = a = 0$, then it induces $b = 0$ by the equations (3.2) – (3.4).

For both cases, it can be concluded that $a = b = c$, which implies that there are two possible solutions which are $a = b = c = 0$ and $a = b = c$, where $a \neq 0$, $b \neq 0$ and $c \neq 0$.

If $a = b = c = 0$, then we have

$$J_1 = xA + yA + zA.$$

If $a = b = c$, where $a \neq 0$, $b \neq 0$ and $c \neq 0$, then we have $a^2 + 3a = 0$. This implies that $a = 0$ or $a = -3$. But $a \neq 0$ in this case, so $a = -3$. Then we have

$$J_2 = (x+3)A + (y+3)A + (z+3)A.$$

Therefor the result holds.

Later, we classify finite-dimensional simple Poisson modules over A annihilated by J via the finite dimensional simple module over $\mathcal{g}(J) := J/_{J^2}$, where J is a Poisson maximal ideal of A.

Theorem 3.1.2 Every finite–dimensional simple Poisson module over A annihilated by J_1 is one-dimensional.

Proof. The Lie algebra $\mathcal{g}(J_1)$ has basis (the images of) x, y, z and bracket $\{x,y\}$, $\{y,z\}$ and $\{z,x\}$ in (3.1) becomes

$$[x,y] = x + y + z, \ \ [y,z] = x + y + z \text{ and } [z,x] = x + y + z.$$

Next we show that $x + y + z$ is in the center of $\mathcal{g}(J_1)$. We have $[x, x+y+z] = [x,x] + [x,y] + [x,z] = 0 + x + y + z - x - y - z = 0$. Similarly, we can also show that $[y, x+y+z] = 0$ and $[z, x+y+z] = 0$. Then $x + y + z$ is in the center of $\mathcal{g}(J_1)$. Moreover, we can see that the derived algebra $[\mathcal{g}(J_1), \mathcal{g}(J_1)]$ is generated by $x + y + z$. Hence the derived algebra $[\mathcal{g}(J_1), \mathcal{g}(J_1)]$ has dimension 1 and is contained in the center of $\mathcal{g}(J_1)$. By the results of Low dimension in section 2, it is isomorphic to the 3-dimensional Heisenberg Lie algebra. By Dixmier (1996) Corollary 1.3.13, every finite–dimensional simple Poisson module over $\mathcal{g}(J_1)$ is one-dimensional and annihilated by

$[\mathscr{g}(J_1), \mathscr{g}(J_1)]$. Therefore, by Theorem 2.13, every finite–dimensional simple Poisson module M over A annihilated by J_1 is one-dimensional.

Theorem 3.1.3 For $d \geq 1$, there is a unique d-dimensional simple Poisson module over A annihilated by J_2.

Proof. For $J_2 = (x+3)A + (y+3)A + (z+3)A$, we consider the Lie algebra $\mathscr{g}(J_2)$.

Let $u = x + 3$, $v = y + 3$ and $w = z + 3$. Then $J_2 = uA + vA + wA$.

Hence, in J_2,

$$\{u, v\} = vu - 2u - 2v + w,$$
$$\{v, w\} = wv + u - 2v - 2w,$$
$$\{w, u\} = uw - 2u + v - 2w.$$

Thus, in $\mathscr{g}(J_2)$,

$$[u, v] = 2u - 2v + w,$$
$$[v, w] = u - 2v - 2w,$$
$$[w, u] = 2u + v - 2w.$$

Next, we will show that these are linearly independent. Let α, β and γ be the scalar such that

$$\alpha[u, v] + \beta[v, w] + \gamma[w, u] = 0.$$

Then $\alpha(2u - 2v + w) + \beta(u - 2v - 2w) + \gamma(2u + v - 2w) = 0$. It induces that

$$2\alpha + \beta + 2\gamma = 0,\ -2\alpha - 2\beta + \gamma = 0 \text{ and } \alpha - 2\beta - 2\gamma = 0.$$

This linear equation system has the augmented matrix $\begin{bmatrix} 2 & 1 & 2 \\ -2 & -2 & 1 \\ 1 & -2 & -2 \end{bmatrix}$, which has non-zero determinant, so the system has one solution that is $\alpha = \beta = \gamma = 0$. It means that $\{[u, v], [v, w], [w, u]\}$ is linearly independent. Hence the derived algebra $[\mathscr{g}(J_2), \mathscr{g}(J_2)]$ has dimension 3, which implies that $\mathscr{g}(J_2) \simeq sl(2, \mathbb{C})$. It is a well-known result, see Henderson (2012) or Humphreys (1972), that, for each $d \geq 1$, $sl(2, \mathbb{C})$ has a unique d-dimensional simple Poisson module annihilated by J_2 and that $sl(2, \mathbb{C})$ is 1-homogeneous. By Theorem 2.13, the result holds for A.

4. CONCLUSION AND DISCUSSION

Studying the Poisson algebra with some complicated bracket could bring the difficulty to find its finite-dimensional simple Poisson modules. In this paper, we use the method shown in Jordan's work (Jordan, 2010). His work tells us that if the Poisson maximal ideals J of A can be identified and the representation theory of the Lie algebra $\mathscr{g}(J)$ is known, then the finite-dimensional simple Poisson A-modules can be found.

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