การมีผลเฉลยของสมการไดโอแฟนไทน์ในรูปแบบ $x^2 + p^2 = y^n$ เมื่อ p เป็นจำนวนเฉพาะ

Solvability of the Diophantine Equation $x^2 + p^2 = y^n$ when p is a Prime

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บทคัดย่อ

จากเราทราบว่า สมการไดโอแฟนไทน์ในรูปแบบ $x^2 + y^2 = z^n$ ไม่มีผลเฉลยที่เป็นจำนวนเต็ม ยกเว้นกรณี $(x, y, z) = (0,0,0)$ เมื่อ n เป็นจำนวนเต็มบวกคู่ที่มากกว่า 2 และสมการรูปแบบนี้มีผลเฉลยที่เป็นจำนวนเต็มเป็นอนันต์ เมื่อ $n = 2$ ซึ่งรู้จักกันเป็นอย่างดีในชื่อของ *“สมการพีทาโกรัส”* ในบทความวิจัยนี้เราจะศึกษาการมีผลเฉลยของสมการในรูปแบบเฉพาะ $x^2 + p^2 = y^n$ เมื่อ p เป็นจำนวนเฉพาะ และ n เป็นจำนวนเต็มบวกคี่ที่มากกว่า 1 โดยประยุกต์สมบัติการแยกตัวประกอบได้เพียงอย่างเดียวในระบบจำนวนเต็มเกาส์เซียนและความรู้พื้นฐานในทฤษฎีจำนวน จากการศึกษาค้นคว้า พบว่าเราสามารถให้เงื่อนไขที่จำเป็นและเพียงพอสำหรับการมีผลเฉลยที่เป็นจำนวนเต็มของสมการ $x^2 + p^2 = y^n$ ในกรณี $n = 3$

ABSTRACT

This paper deals with solving a Diophantine equation in the type of $x^2 + y^2 = z^n$ by using mainly the unique factorization in the ring of Gaussian integers and some elementary facts in number theory. We have known that such an equation has no any integer solution apart from the trivial one when n is an even positive integer greater than 2, and it has known as the “*Pythagorean equation*” in the particular case $n = 2$, which can however be completely solved. Our interest here is to naturally focus on the odd positive one and the unknown variable y (or x) is especially restricted to be a prime number p only. Eventually, it turns out that we are able to give necessary and sufficient conditions for having an integer solution of such an equation in the case $n = 3$.

คำสำคัญ: สมการไดโอแฟนไทน์ โดเมนของการแยกตัวประกอบได้เพียงอย่างเดียว จำนวนเต็มเกาส์เซียน

Keywords: Diophantine equations, Unique factorization domain, Gaussian integers

INTRODUCTION

A Diophantine equation is a polynomial equation in several variables with integer coefficients and the needed solutions are only ordinary integers. Three basic problems concerning a Diophantine equation have arisen naturally as follows: i) is the equation solvable?, ii) if it is solvable, is the number of its solution finite or infinite, and iii) if it is solvable, determine all of its solutions. Here, we will restrict our interest to the problem of the type i) in which there are different approaches to solve such an equation and we may divide them into two groups which are the elementary and advanced methods. The factoring method (or descent), the modular arithmetic method (or local method) and the infinite descent are some main examples for the first group. Some more elementary methods may be seen in the book entitled "*an introduction to Diophantine equations*" written by Andreescu et al. (2010). We note that the word of elementary one means we are applying a property in the integer system only, such as using the unique factorization in the ring of ordinary integers. The other group contains a method involving some advanced technique in the higher rings, namely the Gaussian integers, the quadratic rings and even more general in the ring of integers as seen in the book of Alaca et al. (2004). More precisely, using the unique factorization in Gaussian integers in solving the Diophantine equation $x^2 + y^2 = z^n$ presented by Andreescu et al. (2010), in more general, using the unique factorization in some quadratic rings $\mathbb{Z}[\sqrt{-C}]$ in solving the Diophantine equation $x^2 + C = y^n$, where C is some classes of positive integer may also appearing in the paper of Cohn (1993), and in the most general in this line, using the unique prime ideal factorization in the ring of integers in solving the Diophantine equation in the form of $x^2 + C = y^n$, where $C = 1, 2, 8$, may also found in the paper of Cohn (1993), or even Diophantine equations $px^2 + q^{2m} = y^p$ for some given primes p and q as seen in the work of Muriefah (2008).

Again, in the paper of Cohn (1993), the collection of some known results about answering the solvability of the Diophantine equation $x^2 + C = y^n$, under a condition of an integer C has been investigated. The idea of such results comes from applying the unique factorization or the unique prime ideal factorization in some appropriate ring of integers. Throughout this paper, we will study in case C is a square of some integer and we will be only using the unique factorization in Gaussian integers, normally written as a complex number $a + bi$, where a and b are integers so that we are able to complete the solvability of certain Diophantine equations $x^2 + p^2 = y^n$ for any prime number $\boldsymbol{p}$**.** Note that the condition of a natural number n will be investigated especially the odd case because we have completely known in the book of Andreescu et al. (2010) that the Diophantine equations $x^2 + y^2 = z^n$, where x and y are relatively prime has no integer solution if n is even greater than 2. Actually, there is a general strategy for having an integer solution of general classes of the Diophantine equation $x^2 + y^2 = z^n$ containing in this book. However, we need to work out more mathematically in order to finalize the answer in particular cases as required. In our paper, we will mainly provide necessary and sufficient conditions for having an integer solution of the equations $x^2 + p^2 = y^n$, where p a prime number considered mainly in two cases as $p \equiv 1 (mod\ 4)$ or $p \equiv 3 (mod\ 4)$, and one more trivial case $p = 2$ will be illustrated that it always has an integer solution.

SOME IMPORTANT PROPERTIES IN GUASSIAN INTEGERS

In order to complete our purpose, let us give the necessary and sufficient condition arising from the strategy as mentioned in the previous section for solving the general equation $x^2 + y^2 = z^n$ as follows:

Theorem 1. Let n be an integer greater than 1. Then the equation $x^2 + y^2 = z^n$ with $\gcd(x, y) = 1$ has an integer solution if and only if the equation

$$x + yi = u(a + bi)^n$$

has an integer solution (x, y, a, b) for some $u \in \{\pm 1, \pm i\}$.

The idea how to prove of the above theorem may appear in the page 154-156 of the book written by Andreescu et al. (2010), and the following two lemmas are going to play a crucial role in the line of this proof. In addition, we note that the first lemma below can be actually true for any unique factorization domain, and its analogue in the unique prime ideal factorization has been found in the page 200-202 of the book written by Alaca et al. (2004).

Lemma 2. Let n be any natural number and α, β, γ be non-zero and non-unit Gaussian integers such that $\gcd(\beta, \gamma) = 1$. If $\alpha^n = \beta\gamma$, then there exist β_1, γ_1 and unit elements u, v in Gaussian integers for which

$$\beta = u\beta_1^n \text{ and } \gamma = v\gamma_1^n,$$

where $\gcd(\beta_1, \gamma_1) = 1$.

The proof of the above lemma can be immediately completed by using the unique factorization in the ring of Gaussian integers. Note that for each Gaussian integers β, γ, the meaning of $\gcd(\beta, \gamma) = 1$ is $< \beta, \gamma > = < u >$ for some unit element u in Gaussian integers; that is, the ideal generated by β and γ is the entire ring of Gaussian integers.

The following lemma may be found its proof in the book of Andreescu et al. (2010), especially in the page 154. However, let us show such a proof again in order to understand why integers x and y satisfying the equation $x^2 + y^2 = z^n$ and $\gcd(x, y) = 1$ are very important. The reason is that we need the Gaussian integers $x + yi$ and $x - yi$ to be relatively prime so that we are ready to apply Lemma 2.

Lemma 3. Let n be a natural number such that $n \geq 3$. If the equation $x^2 + y^2 = z^n$ has an integer solution (x, y) with $\gcd(x, y) = 1$, then $x + yi$ and $x - yi$ are coprime.

Proof. Let (x, y, z) be an integer solution to the equation $x^2 + y^2 = z^n$ with $\gcd(x, y) = 1$. We first claim that z is odd. Since $\gcd(x, y) = 1$, so x and y are not both even. If they are both odd, then z is even. This implies that $z^n \equiv 0 (mod\ 8)$ as $n \geq 3$. But indeed, $z^n \equiv 2 (mod\ 8)$, which leads to a contradiction. Hence, we finish the claim. Now, we may factorize the equation $x^2 + y^2 = z^n$ as

$$(x + yi)(x - yi) = z^n \tag{1}$$

Next, we will show that $x + yi$ and $x - yi$ are coprime. Suppose that there exists a Gaussian prime q such that $q|(x - yi)$ and $q|(x + yi)$. Then $q|2\gcd(x, y)$. Clearly, q cannot divide $\gcd(x, y)$. So, let us suppose that $q|2$. Then $q| - i(1 + i)^2$ as $2 = -i(1 + i)^2$. It follows that $q|1 + i$ because q is the Gaussian prime. As $1 + i$ is also a Gaussian prime, we obtain that q and $1 + i$ are associates. Consequently,

$$1+i|x+yi \text{ and } 1+i|x-yi.$$

Thus, by the equation (1), we get $1+i|z^n$. Taking the norm in Gaussian integers, we get $2|z$, a contradiction because z is odd. Hence, we conclude that $x+yi$ and $x-yi$ are coprime, as required. ∎

In particular, if we replace the variable y by a prime number p such that $p\equiv 3(mod\ 4)$, then $\gcd(x,p)=1$ as seen in the first proposition below. Moreover, we will then apply Theorem 1 in order to finalize obtaining the following second proposition.

Proposition 4. Let n be a natural number such that $n\geq 3$, and p be a prime number with $p\equiv 3(mod\ 4)$. If the equation $x^2+p^2=y^n$ has an integer solution (x,y), then $\gcd(x,p)=1$.

Proof. Let (x,y) be an integer solution to the equation in the form written as

$$x^2+p^2=y^n. \tag{2}$$

Suppose that there exists an integer $d\geq 2$ such that $d|x$ and $d|p$. It follows that $d=p$ and so $p|x$. By the equation (2), we get $p|y$. Since $p|x$ and $p|y$, we write $x=pr$ and $y=ps$ for some integers r,s. Thus, the equation (2) becomes the equation $r^2+1=p^{n-2}s^n$. It's a consequence that $p|r^2+1$ as $n\geq 3$. This implies that the equation $x^2\equiv -1(mod\ p)$ has an integer solution. This is impossible because we know that $x^2+1\equiv 0(mod\ p)$ has solution iff $p\equiv 1(mod\ 4)$. Hence $\gcd(x,p)=1$, as desired. ∎

Proposition 5. Let n be a natural number such that $n\geq 3$ and p be a prime number with $p\equiv 3(mod\ 4)$. Then the equation $x^2+p^2=y^n$ has an integer solution if and only if the equation

$$x+pi=u(a+bi)^n$$

has an integer solution (x,a,b) for some $u\in\{\pm 1,\pm i\}$.

Now, we are going to show all solutions depending on two integer parameters of the Diophantine equation $x^2+y^2=z^3$, and the main tool used to obtain such solutions is the unique factorization in the ring Gaussian integers and some facts in elementary number theory.

Theorem 6. The Diophantine equation $x^2+y^2=z^3$ has an integral solution (x,y,z) with $\gcd(x,y)=1$ if and only if there exist integers a,b for which

$$x=a^3-3ab^2, y=3a^2b-b^3 \text{ and } z=a^2+b^2,$$

where $\gcd(a,b)=1$ and $a\not\equiv b(mod\ 2)$.

Proof. Assume that the equation $x^2+y^2=z^3$ has an integer solution (x,y,z) with $\gcd(x,y)=1$. The term on the left-hand side of this equation may be factored as two Gaussian integers like

$$(x+yi)(x-yi)=z^3. \tag{3}$$

Then we can obtain by applying Theorem 1 that $x+yi=u\alpha^3$ for some Gaussian integer α and $u\in\{\pm 1,\pm i\}$. Since $1=(1)^3, -1=(-1)^3, i=(-i)^3$ and $-i=(i)^3$, so the unit factor can be absorbed in the term of α^3.

This implies that there exist integers a,b for which

$$x+yi=(a+bi)^3=(a^3-3ab^2)+(3a^2b-b^3)i.$$

Comparing the real and imaginary parts in the equation above, we get

$$x = a^3 - 3ab^2 \text{ and } y = 3a^2b - b^3.$$

Clearly, $\gcd(a,b) = 1$. If $\gcd(a,b) \neq 1$, then we must have $\gcd(x,y) \neq 1$, which is a contradiction. Now, we have that $a \not\equiv b (mod\ 2)$ because if a and b are either both even or both odd, then $\gcd(a,b) > 1$ or $\gcd(x,y) > 1$, respectively, leading to a contradiction. Note that $x - yi = (a - bi)^3$. Following the equation (3), we eventually obtain that

$$z^3 = ((a+bi)(a-bi))^3 = (a^2+b^2)^3,$$

which means that $z = a^2 + b^2$.

Conversely, let $x = a^3 - 3ab^2$, $y = 3a^2b - b^3$ and $z = a^2 + b^2$ for some integers a, b with the condition that $\gcd(a,b) = 1$ and $a \not\equiv b (mod\ 2)$. Then the integral point (x,y,z) satisfies the considered equation $x^2 + y^2 = z^3$. It remains to show that $\gcd(x,y) = 1$. Suppose that $\gcd(x,y) = d \geq 2$. Then there exists a prime number p such that $p|x$ and $p|y$. Consequently,

$$(p|a \text{ or } p|a^2 - 3b^2) \text{ and } (p|b \text{ or } p|3a^2 - b^2).$$

Then we will divide the consideration into 4 cases:

Case 1: $p|a$ and $p|b$. Then p divides $\gcd(a,b)$, which leads to a contradiction.

Case 2: $p|a$ and $p|3a^2 - b^2$. Then $p|b$, which eventually implies that the $\gcd(a,b) \geq p$, a contradiction.

Case 3: $p|b$ and $p|a^2 - 3b^2$. Following the idea as we did in the case 2, we can reach a contradiction.

Case 4: $p|a^2 - 3b^2$ and $p|3a^2 - b^2$. Then we obtain that $p|2(a^2 + b^2)$. If $p > 2$, then $p|a^2 + b^2$. Again, since $p|3a^2 - b^2$ and $p|a^2 + b^2$, it follows that $p|a$. So, we must have $p|b$. Now, we get $\gcd(a,b) \geq p$, a contradiction. If $p = 2$, then a and b are both even which contradicts the assumption. Hence $\gcd(x,y) = 1$, as desired. ∎

The proof of the above theorem may be seen in the paper of Conrad (2019).

Corollary 7. Let n be a natural number greater than 2 and p be a prime number such that $p \equiv 3 (mod\ 4)$. Then the equation $x^2 + p^2 = y^3$ has an integer solution if and only if

$$x = a^3 - 3ab^2,\ p = 3a^2b - b^3 \text{ and } y = a^2 + b^2$$

for some integers a, b satisfying $\gcd(a,b) = 1$ and $a \not\equiv b (mod\ 2)$.

The above corollary holds by applying Proposition 4 and Theorem 6.

RESULTS ON THE EQUATIONS $x^2 + p^2 = y^3$

Theorem 8. Let p be a prime number such that $p \equiv 3 (mod\ 4)$. Then the Diophantine equation $x^2 + p^2 = y^3$ has an integer solution (x,y) if and only if $p = 3n^2 - 1$ or $p = \sqrt{3n^2 + 1}$ for some non-zero integer n.

Proof. Suppose that there exists a solution (x,y) of the given equation in the theorem above. By Corollary 7, we now have integers a, b satisfying $\gcd(a,b) = 1$ and $a \not\equiv b (mod\ 2)$ for which

$$p = 3a^2b - b^3 = b(3a^2 - b^2). \quad (4)$$

Then $b = \pm 1$ or $b = \pm p$. Let us consider the first case, if $b = \pm 1$, then the equation (4) becomes $\pm p = 3a^2 - 1$. Obviously, it is impossible when $a = 0$. Since $3a^2 - 1 \geq 2$ for all nonzero integer a, so we must have $p = 3a^2 - 1$. For the second case, if we replace $b = \pm p$ in the equation (4), then we have $3a^2 - p^2 = \pm 1$. Since $p \equiv 3(mod\ 4)$, it follows that $3a^2 - p^2 \equiv 0, 3(mod\ 4)$. This leads us to get only $3a^2 - p^2 = -1$, which implies that $p = \sqrt{3a^2 + 1}$, where $a \neq 0$ because of being impossible in the case $a = 0$. Hence, we can finish the proof of this direction.

For the converse direction, if we pick $p = 3n^2 - 1$ for some non-zero integer n or $p = \sqrt{3m^2 + 1}$ for some non-zero integer m, then we may choose the integral point

$$(u, x, a, b) = (1, n^3 - 3n, n, 1) \text{ or } (u, x, a, b) = \left(i, 8m^3 + 3m, \sqrt{3m^2 + 1}, m\right),$$

satisfying the equation $x + pi = u(a + bi)^3$, respectively. By the converse statement of Proposition 5, we can complete the proof of this direction, as desired. ∎

Theorem 9. Let p be a prime number such that $p \equiv 1(mod\ 4)$. Then the Diophantine equation $x^2 + p^2 = y^3$ has an integer solution (x, y) with $\gcd(x, p) = 1$ if and only if $p = \sqrt{3n^2 + 1}$ for some non-zero integer n.

Proof. Let (x, y) be an integer solution of $x^2 + p^2 = y^3$, where $\gcd(x, p) = 1$. By Theorem 6, we obtain that

$$p = 3a^2 b - b^3 = b(3a^2 - b^2). \qquad (5)$$

for some integers a, b such that $\gcd(a, b) = 1$ and $a \not\equiv b(mod\ 2)$. Then $b = \pm 1$ or $b = \pm p$. For the first case, if we plug $b = \pm 1$ in the equation (5), then we have $\pm p = 3a^2 - 1$, which clearly yields $a \neq 0$. It's a consequence that $3a^2 - 1 \geq 2$ for all non-zero integer a. Thus, we would have $p = 3a^2 - 1$. However, since a must be even, so $p = 3a^2 - 1 \equiv -1(mod\ 4)$. This leads to get a contradiction as $p \equiv 1(mod\ 4)$. For the second case, if we replace $b = \pm p$ in the equation (5), then we get $3a^2 - p^2 = \pm 1$. Since a is even, it follows that $p^2 + 1 = 3a^2 \equiv 0(mod\ 4)$. But we have $p^2 + 1 \equiv 2(mod\ 4)$ because $p \equiv 1(mod\ 4)$. Again, here is a contradiction. Thus we must have $3a^2 - p^2 = -1$. Note that $a \neq 0$. Thus, $p = \sqrt{3a^2 + 1}$ for some non-zero integer n. Hence, we can complete the proof of this direction.

For the converse direction, let $p = \sqrt{3n^2 + 1}$ for some non-zero integer n, then selecting the integral point $(a, b) = (n, p)$ leads us to set $x = 8n^3 - 3n$ and $y = 4n^2 + 1$. Obviously, $(p, n) = 1$ as n is always even. So, we immediately get $(p, x) = 1$. By the converse statement of Theorem 6, we can finish the proof of this direction, as desired. ∎

Proposition 10. The Diophantine equation $x^2 + 4 = y^3$ has an integer solution (x, y), where x is odd.

Proof. We may apply Theorem 6 in order to find a solution of the equation $x^2 + 4 = y^3$, where x is odd as follows: Regarding to the theorem, we now choose $a = \pm 1$ and $b = -2$ in order to satisfy

$$2 = b(3a^2 - b^2) = -2(3(\pm 1)^2 - (-2)^2).$$

Consequently,

$$(x, y) = (a(a^2 - 3b^2),\ a^2 + b^2) = (\pm 11, 5),$$

which is the solution of $x^2 + 4 = y^3$, as required. ∎

Notice that if we allow x to be even, then we still yield that the equation $x^2 + 4 = y^3$ is solvable because $(x, y) = (2,2)$. Applying Theorem 8, Theorem 9 and Proposition 10, we immediately get the following two corollaries.

Corollary 11. Let p be a prime number such that $p = 2$ or $p \equiv 3(mod\ 4)$. Then the Diophantine equation $x^2 + p^2 = y^3$ has an integer solution (x, y) if and only if $p = 3n^2 - 1$ or $p = \sqrt{3n^2 + 1}$ for some non-zero integer n.

Corollary 12. Let p be a prime number such that $p = 2$ or $p \equiv 1(mod\ 4)$. Then the Diophantine equation $x^2 + p^2 = y^3$ has an integer solution (x, y) with $\gcd(x, p) = 1$ if and only if $p = \sqrt{3n^2 + 1}$ for some non-zero integer n.

Example 13. Calculating by using the Excel and applying Theorem 8, we obtain the two lists of some prime numbers $p \equiv 3(mod\ 4)$ satisfying the equation $x^2 + p^2 = y^3$ for some integers x, y as illustrated in the following table.

$p = \sqrt{3n^2 + 1}$ and $p < 3650402$	7
$p = 3n^2 - 1$ and $p < 53408$	11,47,107,191,431,587,971,1451,2351,2699,3467,7499, 8111,8747,10091,14699,16427,17327,18251,25391,27647,36299, 41771,44651

To get the list of the primes in the case $p = 3n^2 - 1$ and $p < 53408$, we first use the Excel to show the sequence $a_n = 3n^2 - 1$ and then check together with the list of all primes $p \equiv 3(mod\ 4)$ less than 53408, taken from Sloane (2019). It turns out that there are 24 primes as seen in the second row of the table above. For the other case; that is, $p = \sqrt{3n^2 + 1}$. Then $p^2 - 3n^2 = 1$, which implies that (p, n) is the solution of the Pell's equation $x^2 - 3y^2 = 1$, where p is a prime such that $p \equiv 3(mod\ 4)$. As the fundamental solution of this equation is $(x_1, y_1) = (2,1)$, which actually comes from the infinite simple continued fraction of $\sqrt{3}$, we are able to produce all integer solutions generated by such the fundamental solution of the considered equation at the moment and expressed as the sequence like

$$(x_n, y_n) = \left(\frac{1}{2}\left[(2 + \sqrt{3})^n + (2 - \sqrt{3})^n\right], \frac{1}{2\sqrt{3}}\left[(2 + \sqrt{3})^n - (2 - \sqrt{3})^n\right]\right)$$

for any natural number n. Putting the coefficient 3 in the program in the link of Wolfram (2019) and checking the list of the sequence x_n appearing in the outcome from using the program with $x_n < 3650402$, and it is a prime congruent to 3 modulo 4, we have found that there are only two integral solutions $(p, y) = (7, \pm 2)$ satisfying such a condition as illustrated in the first row of the table above. More detail about how to find a solution of the Pell's equation may be seen in the pages 118-135 appearing in the book of Alaca et al. (2004).

Example 14. Again, calculating by using the Excel and applying Theorem 9 together with following the same recipe like we have just done for the case $p = \sqrt{3n^2 + 1}$ in Example 13, there is only the prime $p = 97$ satisfying the equation $x^2 + p^2 = y^3$ for some integers x, y and $p \equiv 1(mod\ 4)$.

Before ending up this paper, we would like to make two ways of a generalization of our results here as still being in the same line as we did in the paper at the moment: one way is to think about the solvability of the equations $x^2 + p^2 = y^n$, where n is an odd natural number by dividing it into two considerations as $n \equiv 1(mod\ 4)$

or $n \equiv 3 (mod\ 4),$ and the other look is to substitute the position of the prime p in such an equation by another composite number in term of $pq,$ where p and q are different prime numbers and then study their solvability.

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