การเปรียบเทียบผลเฉลยของแบบจำลองพลวัตของก๊าซสองมิติที่มีอุณหภูมิหนึ่งและสองค่า

Comparison of Solutions of One-and Two-Temperature 2D Gas Dynamics Models

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Received: 7 July 2020 | Revised: 18 September 2020 | Accepted: 30 November 2020

บทคัดย่อ

งานวิจัยนี้ได้ศึกษาสมการพลวัตสองมิติของก๊าซที่มีอุณหภูมิแตกต่างกันสองค่า ภายใต้แนวคิดจากสมการลานดาว-เทลเลอร์ โดยการประยุกต์ใช้วิธีการวิเคราะห์กลุ่มในการแก้ปัญหาของระบบสมการดังกล่าว จากการศึกษาเราได้กลุ่มแอดมิทเทดลีหนึ่งกลุ่มและระบบเหมาะที่สุดของพีชคณิตย่อยสองมิติหนึ่งระบบ จากระบบเหมาะสมที่สุดนี้ ทำให้เราได้รูปแบบของผลเฉลยที่ไม่สมมูลกันทุกรูปแบบที่เป็นไปได้ ซึ่งสามารถช่วยในการลดรูประบบสมการเชิงอนุพันธ์ย่อยพลวัตไปสู่ระบบสมการเชิงอนุพันธ์สามัญได้ และจากการวิเคราะห์เชิงลึกทำให้เราทราบผลเฉลยยืนยงของระบบสมการที่เราศึกษา ยิ่งไปกว่านั้นงานวิจัยนี้ยังได้ศึกษาถึงผลการเปรียบเทียบระหว่างผลเฉลยในงานวิจัยกับผลเฉลยแบบดั้งเดิมของสมการพลวัตก๊าซอีกด้วย

ABSTRACT

The two- temperature gas 2D dynamics equations with the Landau-Teller equation are considered in the paper. The group analysis method is applied to the study these equations. An admitted Lie group is found and an optimal system of two-dimensional subalgebras is constructed. Using the optimal system all representations of nonequivalent solutions reducing to a system of ordinary differential equations can be obtained. A detailed analysis of two sets of invariant solutions is given. Comparison of these solutions with solutions of the classical gas dynamics equations is performed in the paper.

คำสำคัญ: ของเหลวภายใต้แรงอัดและก๊าซพลวัต สมการลานดาว-เทลเลอร์ กลุ่มลีและวิธีการพีชคณิต การวิเคราะห์เชิงสมมาตร
Keywords: Compressible fluids and gas dynamics, Landau-Teller equation, Lie group and algebra methods, Symmetry analysis

INTRODUCTION

Phenomena behind strong shock waves include excitation of vibrational modes of the molecules, dissociation, ionization, radiation heat transfer, and other processes. In these flow regions, the translational, rotational and vibrational temperatures of molecules are different and gas is in non- equilibrium state. The relaxation region size can be comparable with the characteristic scale of the problem being solved; therefore, the flow structure is significantly affected by non- equilibrium phenomena. To describe the behavior of gas in the relaxation region, a gas dynamics model involving a system of equations of two-temperature relaxation is used in which vibrational relaxation is described by the inhomogeneous Landau-Teller equation. In the two-dimensional case, the equations describing the mass, momentum and energy balances are (Grigoryev and Ershov, 2017)

$$\begin{aligned}
&\rho_t + u\rho_x + v\rho_y + \rho(u_x + v_y) = 0,\\
&\rho(u_t + uu_x + vu_y) + p_x = 0,\\
&\rho(v_t + uv_x + vv_y) + p_y = 0,\\
&T_t + uT_x + vT_y + (\gamma - 1)T(u_x + v_y) - \gamma_v \frac{(T - T_v)}{\tau} = 0,\\
&T_v + uT_{vx} + vT_{vy} = \frac{(T - T_v)}{\tau}
\end{aligned} \qquad (1)$$

where t is time, u and v are the gas velocities, ρ is the density, T is the static (translational rotational) temperature, T_v is the vibrational temperature, τ is the relaxation time, γ is the adiabatic exponent, γ_v is the exponent of vibrational excitation, the thermodynamic pressure p is given by the state equation

$$p = \rho R_g T$$

where R_g is the gas constant.

The present research is devoted to a comparison of solutions of equations (1) and the classical gas dynamics equations. It should be noted that the analysis of two-dimensional solutions of the studied equations with three independent variables is complicated not only analytically, but also numerically. This is connected with both, the application of numerical schemes and with the variety of initial and boundary conditions. In this sense, ordinary differential equations have several advantages. First, it is easier to derive initial data for a Cauchy problem which can be used in both models. Second, for solving ordinary differential equations without singularities there are well-developed numerical schemes. In particular, in the present paper we apply the sixth-order Runge-Kutta scheme. The comparison is considered using invariant solutions which can be reduced to a system of ordinary differential equations for equations (1) and the two-dimensional gas dynamics equations.

The paper is organized as follows. In the next section, the admitted Lie group of the studied equations is presented. Invariant solutions of the equations are considered in the third section, where numerical solutions are analyzed and compared with corresponding solutions of the classical gas dynamic equations.

ADMITTED LIE GROUP OF EQUATIONS (1)

The classical gas dynamics equations can be considered as a particular case of equations (1) with $\gamma_v = 0$ and $T = T_v$ for the functions ρ, u and T. Complete group analysis of the gas dynamics equations was performed in Ovsiannikov (1978). Analysis of invariant solutions of these equations is given in Ovsiannikov (1994).

For finding invariant solutions of equations (1) one needs to obtain an admitted Lie group. The admitted Lie group can be found by solving the system of determining equations. Calculations show that the admitted Lie group of equations (1) corresponds to the Lie algebra L_7 with the basis generators

$$\begin{gathered} X_1 = \partial_x,\ X_2 = \partial_y,\ X_3 = t\partial_x + \partial_u,\ X_4 = t\partial_y + \partial_V,\ X_5 = v\partial_u - u\partial_v + y\partial_x - x\partial_y, \\ X_6 = \partial_t,\ X_7 = 2T_v\partial_{T_v} + 2T\partial_T - \rho\partial_\rho + v\partial_v + u\partial_u + y\partial_y + x\partial_x. \end{gathered} \tag{2}$$

The set of all substantially different invariant solutions of system (1) can be obtained by using an optimal system of subalgebras (Ovsiannikov, 1993). Our goal is to consider those invariant solutions which can be reduced to the analysis of ordinary differential equations. Such solutions of the two-dimensional equations can be obtained on the basis of two-dimensional subalgebras. The set of all two-dimensional subalgebras of the optimal system of subalgebras of the Lie algebra $L_7$was obtained. It consists of the subalgebras

$$\begin{gathered} \{X_5 + \beta X_6, X_7 + \alpha X_6\},\ \{X_6, X_7 + \alpha X_5\},\ \{X_4 + \alpha X_1, X_7\},\ \{X_2, X_7 + \alpha X_6\},\ \{X_5, X_6\}, \\ \{X_2, X_6 + \alpha X_3 + \beta X_4\},\ \{X_3 + \alpha X_1 + \beta X_2, X_4 + \xi X_1\},\ \{X_4, X_1 + \alpha X_2\},\ \{X_4 + \alpha X_1, X_2\}. \end{gathered} \tag{3}$$

INVARIANT SOLUTIONS

Analysis of the Lie algebra L_7 gives that the generators (2) are also admitted by the classical gas dynamics equations. This property allows us to use the two-dimensional subalgebras (3) also for constructing invariant solutions of the classical gas dynamics equations (Ovsiannikov, 2003; Landau and Lifshitz, 1987). Notice that the invariant solutions corresponding to the subalgebras (3) are reduced to solving systems of ordinary differential equations. For comparison of the solutions of these two models we choose the same initial data for the density, as well as for the velocity. The values of the static temperature and vibrational temperature are variated. For solving the Cauchy problem of the derived systems of ordinary differential equations we apply the sixth-order Runge-Kutta numerical scheme. In this section we only present some of the invariant solutions.

1. The subalgebra $\{X_6, X_7 + \alpha X_5\}$

Analysis of an invariant solution corresponding to this subalgebra depends on α.

Let $\alpha = 0$. The representation of an invariant solution is

$$\rho = \frac{1}{x}R(z),\ U = x\tilde{U}(z),\ V = x\tilde{V}(z),\ T = x^2\tilde{T}(z),\ T_v = x^2\tilde{T}_v(z)\ .$$

This solution of the classical gas dynamics depending on the invariant variable $z = \frac{x}{y}$ is called a conic flow.

Substituting this representation of a solution into equations (1), we obtain the system of the reduced equations

$$\begin{aligned} &R'Vz + (RU)' - V'Rz = 0, \\ &R'\tilde{T}z + \tilde{T}'Rz - \tilde{U}'\left(\tilde{V}Rz^2 + R\tilde{U}z\right) + R\left(\tilde{T} + \tilde{U}^2\right) = 0, \\ &-R'\tilde{T}z^2 - Rz^2\left(\tilde{T}' + \tilde{V}'\tilde{V}\right) + R\tilde{U}\left(\tilde{V}'z + \tilde{V}\right) = 0, \\ &\tilde{T}'\tau\left(\tilde{V}z^2 + \tilde{U}z\right) + \tilde{U}'\tau Tz(\gamma - 1) + \tilde{V}'\tau\tilde{T}z^2(1 - \gamma) + \gamma\tau\tilde{T}\tilde{U} + \gamma_v\left(\tilde{T} - \tilde{T}_v\right) + \tau\tilde{T}\tilde{U} = 0, \end{aligned}$$

$$-\tilde{T}_v{}'\tilde{V}\tau z^2 + \tilde{T}_v{}'\tau\tilde{U}z + 2\tau\tilde{T}_v\tilde{U} - \tilde{T} - \tilde{T}_v = 0. \quad (4)$$

Results of numerical calculations of equations (4) and the gas dynamics equations are presented in Figure 1, where the bold curves correspond to the solution of the classical gas dynamics equations; the dashed curve is the vibrational temperature, and the continuous curves correspond to the solution of equations (4). On the graphs of temperatures, it is seen that because of energy storage in the vibrational mode, the static (translational) temperature evolves more slowly than the temperature in an ideal gas. Corresponding to the higher temperature, the density in vibrationaly excited gas is lower than that one in an ideal gas.

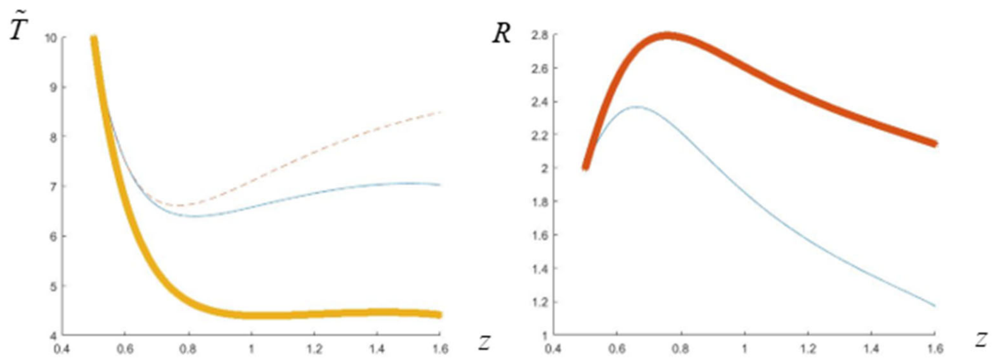


Figure 1. The left figure corresponds to the temperatures $\tilde{T}_v$ and $\tilde{T}$ of the classical gas dynamics equations and of system (1). The right figure corresponds to the density. Initial data are $R(0.5) = 2, \tilde{U}(0.5) = 1, \tilde{V}(0.5) = 0, \tilde{T}(0.5) = 10, \tilde{T}_v(0.5) = 10.$

Consider the second case when $\boldsymbol{\alpha \neq 0}$. In this case it is more convenient to rewrite equations (1) in polar coordinate system. A representation of the invariant solutions becomes $\rho = \frac{1}{r}R(z)$, $U = r\tilde{U}(z)$, $V = r\tilde{V}(z)$, $T = r^2\tilde{T}(z)$, $T_v = r^2\tilde{T}_v(z)$, $z = re^{\frac{-\theta}{\alpha}}$.

Substitution of the latter representation of a solution into equations (1) leads to the system of reduced equations

$$R' = \frac{-\alpha R\tilde{U}'z + RV'z - \alpha R\tilde{U}}{z(\alpha\tilde{U} - \tilde{V})}, \tilde{T}_v{}' = \frac{\alpha(\tilde{T} - 2\tilde{T}_v\tilde{U} - \tilde{T}_v)}{z(\alpha\tilde{U} - \tilde{V})}$$

$$\tilde{T}' = \frac{\alpha\tilde{T}\tilde{U}'z(1-\gamma) + \tilde{T}\tilde{V}'z(\gamma - 1) + \alpha(-2\gamma\tilde{T}\tilde{U} - \gamma_v(\tilde{T} + \tilde{T}_v))}{z(\alpha\tilde{U} - \tilde{V})}$$

$$\tilde{U}'z\left(R\alpha^2\gamma\tilde{T} + \alpha^2\tilde{U}^2 - 2\alpha\tilde{U}\tilde{V} + \widetilde{V^2}\right) + \alpha R\gamma\tilde{T}\tilde{V}'z + \alpha R\left(-2\alpha\gamma\tilde{T}\tilde{U} - \alpha\gamma_v(\tilde{T} + \tilde{T}_v) - \tilde{T}\tilde{V} + \alpha\left(\alpha\tilde{U}^3 - \alpha\tilde{U}\tilde{V}^2 - \tilde{V}\tilde{U}^2 + \tilde{V}^3\right)\right) = 0$$

$$\alpha\gamma R\tilde{T}\tilde{U}'z - \gamma R\tilde{T}\tilde{V}'z + \tilde{V}'z(\alpha^2\tilde{U}^2 - 2\alpha\tilde{U}\tilde{V} + \tilde{V}^2) + \alpha R(2\gamma\tilde{T}\tilde{U}) + \gamma_v(\tilde{T} + \tilde{T}_v) + \tilde{T}\tilde{U}) + 2\alpha\tilde{U}\tilde{V}(\alpha\tilde{U} - \tilde{V}) = 0. \quad (5)$$

The results of numerical calculations of equations (5) and the gas dynamics equations are presented in Figure 2, where the notations are similar to in Figure 1. It should be mentioned here that the presented solutions are substantially two-dimensional.

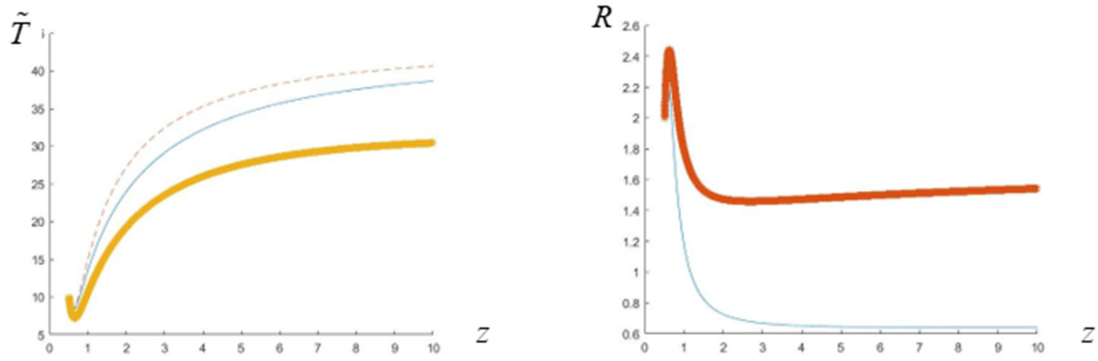


Figure 2. The left figure corresponds to the temperatures $\tilde{T}_\nu$ and $\tilde{T}$ of the gas and of system (1).

The right figure corresponds to density. Initial data are $R(0.5) = 2, \tilde{U}(0.5) = 1, \tilde{V}(0.5) = 0, \tilde{T}(0.5) = 10, \tilde{T}_\nu(0.5) = 10, \alpha$=1.

2. The subalgebra $\{X_5 + \beta X_6, X_7 + \alpha X_6\}$

The analysis of invariant solutions corresponding to this subalgebra has to be separated into two cases: $\alpha \neq 0$ and $\alpha = 0$.

Let $\alpha \neq 0$. The representation of an invariant solution is

$$\rho = r^{-1}R(z),\ U = r\tilde{U}(z),\ V = r\tilde{V}(z),\ T = r^2\tilde{T}(z),\ T_\nu = r^2\tilde{T}_\nu(z),\ z = re^{\frac{t-\beta\theta}{\alpha}}.$$

The system of reduced equations becomes

$$-R\beta\tilde{V}z + R'\alpha z(\tilde{U}+1) + Rz(\alpha\tilde{U} - \beta\tilde{V}') + \alpha R\tilde{U} = 0,$$
$$\alpha Rz(R\tilde{T})' + \beta R\widetilde{U}'\tilde{V}z + \alpha R\widetilde{U}'z(\tilde{U}+1) + \alpha R(\tilde{V}^2 - R\tilde{T} - \tilde{U}^2) = 0,$$
$$\beta Rz((R\tilde{T})' - \beta R\tilde{V}'\tilde{V}z + \alpha R\tilde{V}'z(\tilde{U}+1) + 2\alpha R\tilde{U}\tilde{V} = 0,$$
$$-\beta\tilde{T}'\tilde{V}z + \alpha\tilde{T}'z(\tilde{U}+1) + \alpha\tilde{T}\tilde{U}'z(\gamma-1) + \beta\tilde{T}\tilde{V}'z(1-\gamma) + \alpha\left(2\gamma\tilde{T}\tilde{U} + \gamma_\nu(\tilde{T} - \tilde{T}_\nu)\right) = 0,$$
$$-\beta\tilde{T}_\nu{}'\tilde{V}z + \alpha\widetilde{T_\nu}'z(\tilde{U}+1) + \alpha(-\tilde{T} + 2\tilde{T}'_\nu U + \tilde{T}_\nu) = 0.$$

Results of the numerical calculations are presented in Figure 3.

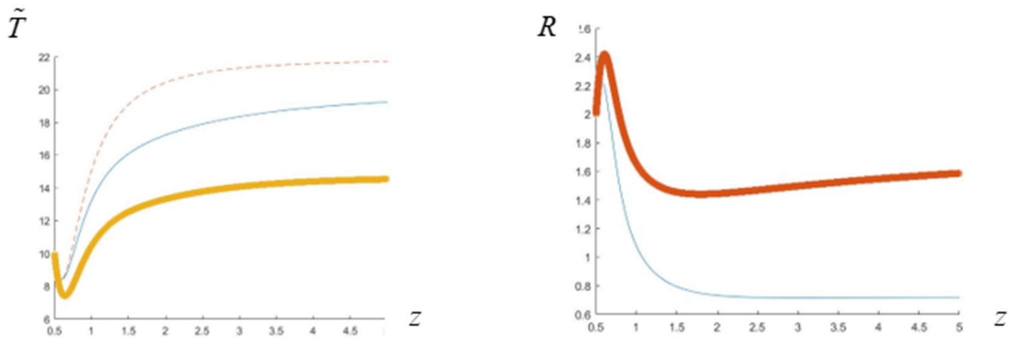


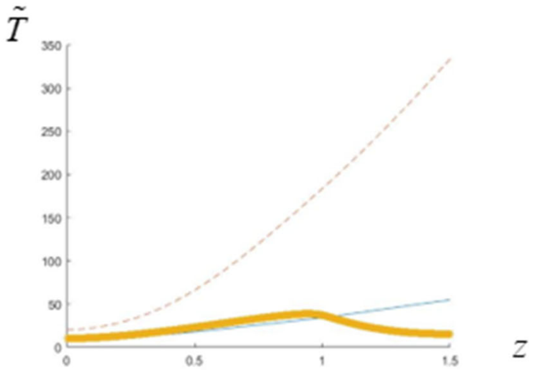
Figure 3. The left figure corresponds to the temperatures $\tilde{T}_\nu$ and $\tilde{T}$ in gas and system (1).

The right figure corresponds density initial data are $R(0.5) = 2, \tilde{U}(0.5) = 1, \tilde{V}(0.5) = 0, \tilde{T}(0.5) = 10, \tilde{T}_\nu(0.5) = 10, \alpha$=1, β=1.

Consider $\alpha = 0$. The representation of an invariant solution has the form $\rho = r^{-1}R(z),\ U = r\tilde{U}(z),\ V = r\tilde{V}(z),\ T = r^2\tilde{T}(z),\ T_\nu = r^2\tilde{T}_\nu(z), z = t - \beta\theta$. Substituting this representation into equations (1), we obtain the system of reduced equations

$$R'(1-\beta\tilde{V}) - R(\beta\tilde{V}' - \tilde{U}) = 0,$$
$$\tilde{U}'(1-\beta\tilde{V}) - \tilde{V} + R\tilde{T} + \tilde{U} = 0,$$
$$\beta R(R\tilde{T})' - R\tilde{V}'(\beta\tilde{V} - 1) + 2R\tilde{U}\tilde{V} = 0,$$
$$\tilde{T}'(\beta\tilde{V} + 1) + \beta\tilde{T}\tilde{V}'(1-\gamma) + 2\gamma\tilde{T}\tilde{U} + \gamma_v(\tilde{T} - \tilde{T}_v) = 0,$$
$$\tilde{T}'_v(\beta\tilde{V} + 1) + \tilde{T} + \tilde{T}_v(2\tilde{U} + 1) = 0.$$

Results of the numerical calculations are presented in Figure 4.



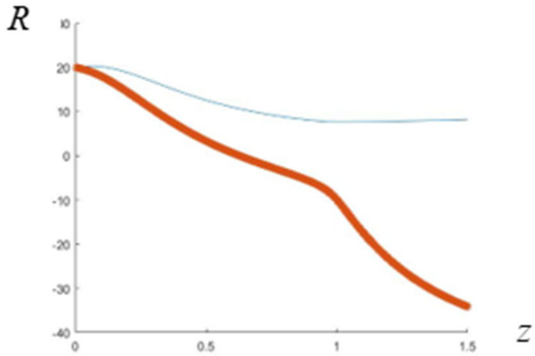


Figure 4. The left figure corresponds to the temperatures $\tilde{T}_v$ and $\tilde{T}$ of the gas and of system (1). The right figure corresponds density. Initial data are $R(0) = 20, \tilde{U}(0) = 0, \tilde{V}(0) = 2, \tilde{T}(0) = 10, \tilde{T}_v(0) = 20, \beta$=1.

CONCLUSIONS

A comparison of solutions of two models is presented in the paper. The classical gas dynamics equations and the gas dynamics equations with the Landau-Teller equation were considered. The reduction of the studied equations to a system of ordinary differential equations allows us to make comparisons of these two models for different parameters. The numerical experiments show that, at least for the invariant solutions considered here, the relaxation process (1) affects the gas evolution in an essential manner when compared with an ideal gas under the same initial data**.**

ACKNOWLEDGMENTS

This research was supported by Mae Fah Luang University.

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