ปัญหาของการเลื่อนข้อมูลของสมการเชิงอนุพันธ์โดยใช้การแปลงเชิงปริพันธ์
The Shifted Data Problems of Differential Equations by Using Integral Transforms

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บทคัดย่อ

บทความนี้พิจารณาการหาผลเฉลยของสมการเชิงอนุพันธ์ซึ่งมีการเลื่อนของข้อมูลเงื่อนไขขอบ โดยใช้การแปลงลาปลาซและการแปลงเอลซากิ ซึ่งในบทความจะใช้คุณสมบัติของ $L\{f^{(n)}(t)\}$ และ $E[f^{(n)}(t)]$ โดยที่ $f^{(i)}(k)=\beta_i$ สำหรับจำนวนจริงบวก k และ $i=0,1,2,...,n-1$ ผลการศึกษาพบว่าวิธีการดังกล่าวสามารถหาผลเฉลยสมการเชิงอนุพันธ์ได้และสะดวกขึ้น

ABSTRACT

In this article, we have checked the shifted data problems by Laplace transform and Elzaki transform. We put emphasis on the representation of $L\{f^{(n)}(t)\}$ and $E[f^{(n)}(t)]$ to $f^{(i)}(k)=\beta_i$ for any positive number k and $i=0,1,2,...,n-1$. Results of this research yields an easy method to find solutions to differential equations.

คำสำคัญ: การแปลงลาปลาซ การแปลง Elzaki การเลื่อนข้อมูล สมการเชิงอนุพันธ์
Keywords: Laplace transform, Elzaki transform, Shifted Data, Differential Equations

INTRODUCTION

The Laplace transform and the Elzaki transform are powerful integral transforms which can be used in some cases to solve linear differential equations with given initial conditions. The Laplace transforms of derivatives of $f(t)$ are represented by

$$L\left\{f^{(n)}(t)\right\} = s^n L\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0). \tag{1}$$

In the same way, Elzaki transforms of derivatives of $f(t)$ are represented by

$$E[f^{(n)}(t)] = \frac{E[f(t)]}{u^n} - \sum_{k=0}^{n-1} u^{2-n+k} f^{(k)}(0). \tag{2}$$

In both transformations, almost all initial value problems are define with ones at zero (Elzaki, 2011). We would like to check the shifted data problems which are related to integral transforms (Kim, 2014).

MATERIALS AND METHODS

In this section, we introduce the definition and basic properties of the Laplace transform and the Elzaki transform.

1. The Laplace Transform

Suppose that f is a real-valued function of the variable $t > 0$ and s is a real parameter. We define the Laplace transform of f as

$$F(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = \lim_{\tau\to\infty}\int_0^\tau e^{-st} f(t)dt \tag{3}$$

whenever the limit exists (Johar, 2019).

Next, we introduce some properties of the Laplace transform which are often used in this research.

1.1 The Linearity Property of the Laplace Transform

Linearity of the Laplace transform is an important result which states that

$$L\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 L\{f_1(t)\} + c_2 L\{f_2(t)\}, \tag{4}$$

where f_1, f_2 are real-valued functions whose Laplace transform exists and c_1, c_2 are arbitrary constants.

1.2 Derivatives of the Laplace Transform

For all positive integers n, the n^{th} derivative of the Laplace transform is given by

$$L\{f^{(n)}(t)\} = s^n L\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0), \tag{5}$$

where $f,\ f',\ f'',\ldots,\ f^{(n-1)}$ are continuous functions on $[0,\infty)$.

1.3 The Inverse of the Laplace Transform

Given a function $F(s)$, if we can find a function $f(t)$ such that $L\{f(t)\} = F(s)$, then the inverse Laplace transform is denoted by

$$L^{-1}\{F(s)\} = f(t), \quad t \geq 0 . \tag{6}$$

Note that the inverse Laplace transform is linear, that is,

$$L^{-1}\{c_1 F_1(s) + c_2 F_2(s)\} = c_1 L^{-1}\{F_1(s)\} + c_2 L^{-1}\{F_2(s)\}. \tag{7}$$

This follows from the linearity of L and holds in the domain common to F_1 and F_2, where c_1, c_2 are arbitrary constants (Joel L, 1988).

2. The Elzaki Transform

The Elzaki transform is defined for functions of exponential order. We consider a function in the set A defined by

$$A = \left\{ f(t) : \exists M, k_1 \ and \ k_2 > 0 : \left|f(t)\right| < Me^{\frac{|t|}{k_j}}, \ if \ t \in (-1)^j \times [0,\infty) \right\}. \tag{8}$$

According to Elzaki (2011), the Elzaki transform is defined by

$$E[f(t)] = u\int_0^\infty f(t) e^{-\frac{t}{u}} dt, \quad t \geq 0 \quad \text{and} \quad k_1 \leq u \leq k_2 . \tag{9}$$

2.1 The Linearity Property of the Elzaki Transform

Linearity of the Elzaki transform is an important result which states that

$$E\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 E\{f_1(t)\} + c_2 E\{f_2(t)\}, \tag{10}$$

where c_1, c_2 are arbitrary constants (Sudhanshu, 2018).

2.2 Derivatives of the Elzaki Transform

For all positive integers n, the Elzaki transform of the n^{th} derivative of function can be given as

$$E[f^{(n)}(t)] = \frac{E[f(t)]}{u^n} - \sum_{k=0}^{n-1} u^{2-n+k} f^{(k)}(0). \tag{11}$$

(Saad, 2019).

2.3 The Inverse of the Elzaki Transform

Given a function $T(u)$, if we can find a function $f(t)$ such that $E[f(t)] = T(u)$, then the inverse Elzaki transform is denoted by

$$E^{-1}[T(u)] = f(t). \tag{12}$$

Note that the inverse Elzaki transform is linear, that is,

$$E^{-1}\left[c_1 T_1(u) + c_2 T_2(u)\right] = c_1 E^{-1}\left[T_1(u)\right] + c_2 E^{-1}\left[T_2(u)\right], \tag{13}$$

where c_1, c_2 are arbitrary constants (Sudhanshu, 2018).

Table 1. The Inverse Elzaki Transform of Some Elementary Functions

$T(u)$	$E^{-1}[T(u)]$
$u^{n+2},\ n \in N$	$\frac{t^n}{n!}$
$u^{n+2},\ n > -1$	$\frac{t^n}{\Gamma(n+1)}$
$\frac{u^2}{1-au}$	e^{at}
$\frac{u^3}{1+a^2u^2}$	$\frac{\sin at}{a}$
$\frac{u^2}{1+a^2u^2}$	$\cos at$

RESULTS

Theorem 1. Let $t = t_1 + k$ for $k > 0$. Then the solution of the shifted data problem

$$y^{(n+1)}(t) + a_n y^{(n)}(t) + a_{n-1} y^{(n-1)}(t) + \cdots + a_1 y'(t) + a_0 y(t) = r(t) \tag{14}$$

with condition $y^{(i)}(k) = b_{n-i}$ for $i = 0, 1, 2, ..., n$ has the form

$$y(t) = L^{-1}\{Y_1\}\Big|_{t_1 \to t-k},$$

where

$$Y_1 = \frac{L\{r(t_1+k)\} + b_0 + b_1(s+a_n) + b_2(s^2 + a_n s + a_{n-1}) + \cdots + b_{n-1}(s^{n-1} + \sum_{i=2}^{n} a_i s^{i-2}) + b_n(s^n + \sum_{i=1}^{n} a_i s^{i-1})}{s^{n+1} + \sum_{i=0}^{n} a_i s^i} \tag{15}$$

and $L\{y_1(t_1)\} = Y_1,\ y_1(t_1) = y(t)$.

Proof Let $t=t_1+k$ for $k>0$. Then Eq. (14) becomes

$$y_1^{(n+1)}(t_1)+a_n y_1^{(n)}(t_1)+a_{n-1}y_1^{(n-1)}(t_1)+\cdots+a_1 y_1'(t_1)+a_0 y_1(t_1)=r(t_1+k) \tag{16}$$

and $y_1^{(i)}(0)=b_{n-i}$ for $y_1(t_1)=y(t)$. Taking Laplace transforms in Eq. (16), we have

$$L\{y_1^{(n+1)}(t_1)\}+a_n L\{y_1^{(n)}(t_1)\}+a_{n-1}L\{y_1^{(n-1)}(t_1)\}+\cdots+a_1 L\{y_1'(t_1)\}+a_0 L\{y_1(t_1)\}=L\{r(t_1+k)\}. \tag{17}$$

Applying Eq. (5) to Eq. (17), we obtain

$$\begin{aligned}&\left(s^{n+1}L\{y_1(t_1)\}-s^n y_1(0)-s^{n-1}y_1'(0)-\cdots-sy_1^{(n-1)}(0)-y_1^{(n)}(0)\right)\\&+a_n\left(s^n L\{y_1(t_1)\}-s^{n-1}y_1(0)-s^{n-2}y_1'(0)-\cdots-sy_1^{(n-2)}-y_1^{(n-1)}(0)\right)\\&+a_{n-1}\left(s^{n-1}L\{y_1(t_1)\}-s^{n-2}y_1(0)-s^{n-3}y_1'(0)-\cdots-sy_1^{(n-3)}-y_1^{(n-2)}(0)\right)\\&\vdots\\&+a_2\left(s^2 L\{y_1(t_1)\}-sy_1(0)-y_1'(0)\right)+a_1\left(sL\{y_1(t_1)\}-y_1(0)\right)+a_0 L\{y_1(t_1)\}=L\{r(t_1+k)\}.\end{aligned} \tag{18}$$

Rearranging Eq. (18), we have

$$\begin{aligned}&L\{y_1(t_1)\}\left[s^{n+1}+a_n s^n+a_{n-1}s^{n-1}+\cdots+a_2 s^2+a_1 s+a_0\right]\\&-y_1(0)\left[s^n+a_n s^{n-1}+a_{n-1}s^{n-2}+\cdots+a_3 s^2+a_2 s+a_1\right]\\&-y_1'(0)\left[s^{n-1}+a_n s^{n-2}+a_{n-1}s^{n-3}+\cdots+a_4 s^2+a_3 s+a_2\right]\\&-y_1''(0)\left[s^{n-2}+a_n s^{n-3}+a_{n-1}s^{n-4}+\cdots+a_5 s^2+a_4 s+a_3\right]\\&-\cdots-y_1^{(n-2)}(0)\left[s^2+a_n s+a_{n-1}\right]-y_1^{(n-1)}(0)\left[s+a_n\right]-y_1^{(n)}(0)=L\{r(t_1+k)\}.\end{aligned} \tag{19}$$

Substituting $y_1^{(i)}(0)=b_{n-i}$ and $L\{y_1(t_1)\}=Y_1$ in Eq. (19) leads to

$$Y_1=\frac{L\{r(t_1+k)\}+b_0+b_1(s+a_n)+b_2(s^2+a_n s+a_{n-1})+\cdots+b_{n-1}(s^{n-1}+\sum_{i=2}^{n}a_i s^{i-2})+b_n(s^n+\sum_{i=1}^{n}a_i s^{i-1})}{s^{n+1}+a_n s^n+a_{n-1}s^{n-1}+\cdots+a_2 s^2+a_1 s+a_0}. \tag{20}$$

Taking the inverse Laplace transform in Eq. (20), we have $y_1(t_1)=L^{-1}\{Y_1\}$. Since $t=t_1+k$, the solution of Eq. (14) is

$$y(t)=L^{-1}\{Y_1\}\Big|_{t_1\to t-k}$$

for $L\{y_1(t_1)\}=Y_1$ and $y_1(t_1)=y(t)$.

Theorem 2 Let $t=t_1+k$ for $k>0$. Then the solution of the shifted data problem

$$y^{(n+1)}(t)+a_n y^{(n)}(t)+a_{n-1}y^{(n-1)}(t)+\cdots+a_1 y'(t)+a_0 y(t)=r(t) \tag{21}$$

with condition $y^{(i)}(k)=b_{n-i}$ for $i=0,1,2,\ldots,n$ has the form

$$y(t)=E^{-1}[Y_1]\Big|_{t_1\to t-k}$$

where

$$Y_1=\frac{E[r(t_1+k)]+b_0 u+b_1(1+a_n u)+b_2(u^{-1}+a_n u^0+a_{n-1}u)+\cdots+b_{n-1}(u^{2-n}+\sum_{i=2}^{n}a_i u^{3-i})+b_n(u^{2-n}+\sum_{i=1}^{n}a_i u^{2-i})}{\frac{1}{u^{n+1}}+a_0+\frac{a_1}{u^1}+\frac{a_2}{u^2}+\cdots+\frac{a_{n-2}}{u^{n-2}}+\frac{a_{n-1}}{u^{n-1}}+\frac{a_n}{u^n}}. \tag{22}$$

Proof Let $t=t_1+k$ for $k>0$. Then Eq. (21) becomes

$$y_1^{(n+1)}(t_1)+a_n y_1^{(n)}(t_1)+a_{n-1}y_1^{(n-1)}(t_1)+\cdots+a_1 y_1'(t_1)+a_0 y_1(t_1)=r(t_1+k) \tag{23}$$

and $y_1^{(i)}(0)=b_{n-i}$ for $y_1(t_1)=y(t)$. Taking Elzaki transforms in Eq. (23), we have

$$E\{y_1^{(n+1)}(t_1)\}+a_n E\{y_1^{(n)}(t_1)\}+a_{n-1}E\{y_1^{(n-1)}(t_1)\}+\cdots+a_1 E\{y_1'(t_1)\}+a_0 E\{y_1(t_1)\}=E\{r(t_1+k)\}. \tag{24}$$

Applying Eq. (11) in Eq. (24), we obtain

$$\frac{E[y_1(t_1)]}{u^{n+1}}-\sum_{k=0}^{n}u^{1-n+k}y^{(k)}(0)+a_n\left(\frac{E[y_1(t_1)]}{u^n}-\sum_{k=0}^{n-1}u^{2-n+k}y^{(k)}(0)\right)+a_{n-1}\left(\frac{E[y_1(t_1)]}{u^{n-1}}-\sum_{k=0}^{n-2}u^{3-n+k}y^{(k)}(0)\right)+\cdots$$
$$+a_2\left(\frac{E[y_1(t_1)]}{u^2}-y(0)-uy'(0)\right)+a_1\left(\frac{E[y_1(t_1)]}{u}-uy(0)\right)+a_0E[y_1(t_1)]=E[r(t_1+k)]. \quad (25)$$

Rearranging Eq. (25), we have

$$E[y_1(t_1)]\left(\frac{1}{u^{n+1}}+\frac{a_n}{u^n}+\frac{a_{n-1}}{u^{n-1}}+\frac{a_{n-2}}{u^{n-2}}+\cdots+\frac{a_2}{u^2}+\frac{a_1}{u}+a_0\right)-y(0)\left(u^{1-n}+a_nu^{2-n}+a_{n-1}u^{3-n}+\cdots+a_2+a_1u\right)$$
$$-y'(0)\left(u^{2-n}+a_nu^{3-n}+a_{n-1}u^{4-n}+\cdots+a_3+a_2u\right)\quad -y''(0)\left(u^{3-n}+a_nu^{4-n}+a_{n-1}u^{5-n}+\cdots+a_4+a_3u\right)-\cdots$$
$$-y^{(n-2)}(0)\left(u^{-1}+a_n+a_{n-1}u\right)-y^{(n-1)}(0)\left(u^0+a_nu\right)-y^{(n)}(0)\left(u\right)=E[r(t_1+k)]. \quad (26)$$

Substituting $y_1^{(i)}(0)=b_{n-i}$ and $E[y_1(t_1)]=Y_1$ in Eq. (26) leads to

$$Y_1=\frac{E[r(t_1+k)]+b_0u+b_1(1+a_nu)+b_2(u^{-1}+a_nu^0+a_{n-1}u)+\cdots+b_{n-1}(u^{2-n}+\sum_{i=2}^{n}a_iu^{3-i})+b_n(u^{2-n}+\sum_{i=1}^{n}a_iu^{2-i})}{\frac{1}{u^{n+1}}+a_0+\frac{a_1}{u^1}+\frac{a_2}{u^2}+\cdots+\frac{a_{n-2}}{u^{n-2}}+\frac{a_{n-1}}{u^{n-1}}+\frac{a_n}{u^n}}. \quad (27)$$

Taking the inverse Elzaki transform in Eq. (27), we have $y_1(t_1)=E^{-1}[Y_1]$. Since $t=t_1+k$, the solution of Eq. (21) is

$$y(t)=E^{-1}[Y_1]\big|_{t_1\to t-k}$$

for $E[y_1(t_1)]=Y_1$ and $y_1(t_1)=y(t)$.

APPLICATIONS

In this section we apply the Laplace transform and the Elzaki transform to solve some differential equations.

Example 1. Using the Laplace transform to solve the differential equation

$$y'''+3y''-y'-3y=0 \quad (28)$$

with $y(3)=1,\ y'(3)=1$ and $y''(3)=-1$.

Solution. From Eq. (28) and Theorem 1, we have $n=2,\ k=3$.

Substituting these in Eq. (15), we have

$$Y_1=\frac{L\{r(t_1+3)\}+b_0+b_1(s+a_2)+b_2(s^2+a_2s+a_{2-1})}{s^{2+1}+\sum_{i=0}^{2}a_is^i}. \quad (29)$$

Next, we substitute $a_2=3,\ a_1=-1,\ a_0=-3,\ b_2=1,\ b_1=1,\ b_0=-1$ and $L\{r(t_1+3)\}=0$ in Eq. (29). Then

$$Y_1=\frac{s^2+4s+1}{s^3+3s^2-s-3}\ . \quad (30)$$

By Theorem 1, the solution of Eq. (28) is

$$y(t)=L^{-1}\left\{\frac{s^2+4s+1}{s^3+3s^2-s-3}\right\}\Bigg|_{t\to t-3} \quad (31)$$
$$=L^{-1}\left\{\frac{3}{4(s-1)}+\frac{1}{2(s+1)}-\frac{1}{4(s+3)}\right\}\Bigg|_{t_1\to t-3}$$

$$=\left[\frac{3}{4}e^{t_1}+\frac{1}{2}e^{-t_1}-\frac{1}{4}e^{-3t_1}\right]\Bigg|_{t_1\to t-3}.$$

Hence $y(t)=\frac{3}{4}e^{t-3}+\frac{1}{2}e^{-t+3}-\frac{1}{4}e^{-3t+9}$.

Example 2. Using the Laplace transform to solve the differential equation

$$y''+y=2t \tag{32}$$

with $y\left(\frac{\pi}{4}\right)=\frac{\pi}{4}$ and $y'\left(\frac{\pi}{4}\right)=2-\sqrt{2}$.

Solution. From Eq. (32) and Theorem 1, we have $n=1,\ k=\frac{\pi}{4}$.

Substituting these in Eq. (15), we have

$$Y_1=\frac{L\left\{r\left(t_1+\frac{\pi}{4}\right)\right\}+b_0+b_1(s+a_1)}{s^{1+1}+\sum_{i=0}^{1}a_is^i}. \tag{33}$$

Next, we substitute $a_1=0,\ a_0=1,\ b_1=\frac{\pi}{4},\ b_0=2-\sqrt{2}$ and $L\left\{r\left(t_1+\frac{\pi}{4}\right)\right\}=2t_1+\frac{\pi}{2}$ in Eq. (33). Then

$$Y_1=\frac{8+2\pi s+4s^2(2-\sqrt{2})+\pi s^3}{4s^2(s^2+1)}. \tag{34}$$

By Theorem 1, the solution of Eq. (32) is

$$y(t)=L^{-1}\left\{\frac{8+2\pi s+4s^2(2-\sqrt{2})+\pi s^3}{4s^2(s^2+1)}\right\}\Bigg|_{t\to t-\frac{\pi}{4}} \tag{35}$$

$$=L^{-1}\left\{\frac{\pi}{2s}+\frac{2}{s^2}-\frac{\pi s}{4(s^2+1)}-\frac{\sqrt{2}}{s^2+1}\right\}\Bigg|_{t_1\to t-\frac{\pi}{4}}$$

$$=\left[\frac{\pi}{2}+2t_1-\frac{\pi}{4}\cos t_1-\sqrt{2}\sin t_1\right]\Bigg|_{t_1\to t-\frac{\pi}{4}}.$$

Hence

$$y(t)=2t_1-\frac{\pi}{4}\cos\left(t_1-\frac{\pi}{4}\right)-\sqrt{2}\sin\left(t_1-\frac{\pi}{4}\right).$$

Example 3. Using the Elzaki transform to solve the differential equation

$$y'''+y''=6(t-1)^2+4 \tag{36}$$

with $y(1)=0,\ y'(1)=0$ and $y''(1)=0$.

Solution. From Eq. (36) and Theorem 2, we have $n=2,\ k=1$.

Substituting these in Eq. (22), we have

$$Y_1=\frac{E[r(t_1+1)]+b_0u+b_1(1+a_nu)+b_2(u^{-1}+a_nu^0+a_{n-1}u)}{\frac{1}{u^{2+1}}+a_0+\frac{a_1}{u^1}+\frac{a_2}{u^2}} \tag{37}$$

Next, we substitute $a_2 = 1,\ a_1 = 0,\ a_0 = 0,\ b_2 = 0,\ b_1 = 0,\ b_0 = 0$ and $E[r(t_1+1)] = 12u^4 + 4u^2$ in Eq. (37). Then

$$Y_1 = \frac{12u^4 + 4u^2 + (0)u + (0)(1+1u) + (0)(u^{-1} + 1u^0 + 0u)}{\frac{1}{u^3} + 0 + \frac{0}{u} + \frac{1}{u^2}} \tag{38}$$

By Theorem 2, the solution of Eq. (36) is

$$\begin{aligned} y(t) &= E^{-1}\left[\frac{12u^4 + 4u^2}{\frac{1}{u^3} + \frac{1}{u^2}}\right]\Bigg|_{t_1 \to t-1} \\ &= E^{-1}\left[\frac{12u^7 + 4u^5}{u+1}\right]\Bigg|_{t_1 \to t-1} \\ &= E^{-1}\left[12u^6 - 12u^5 + 16u^4 - 16u^3 + 16u^2 - \frac{16u^2}{u+1}\right]\Bigg|_{t_1 \to t-1} \\ &= \left[\frac{1}{2}t_1^4 - 2t_1^3 + 8t_1^2 - 16t_1 + 16 - 16e^{-t_1}\right]\Bigg|_{t_1 \to t-1}. \end{aligned}$$

Hence $y(t) = \frac{1}{2}(t-1)^4 - 2(t-1)^3 + 8(t-1)^2 - 16(t-1) + 16 - 16e^{-(t-1)}$.

CONCLUSIONS

In this study, the applications of the Laplace transform and the Elzaki transform to the solution of differential equations with constant coefficients have been demonstrated.

REFERENCES

Aatqb, J. M. A. (2019). Notes on Laplace Transforms. https://www.researchgate.net/publication/333894393_Notes_on_the_Laplace_Transforms (Accessed: January 2019).

Aggarwal, S., Singh, D. P., Asthana, N. Gupta. A. R. (2018). Application of Elzaki Transform for Solving Population Growth and Decay Problems. Journal of Emerging Technologies and Innovative Research. 5(9): 281-284.

Elzaki, T. M. and Ezaki, S. M. (2011). On the ELzaki Transform and Higher Order Ordinary Differential Equations. Advances in Theoretical and Applied Mathematics 6(1): 107-113.

Elzaki, T. M. and Ezaki, S. M. (2011). On the ELzaki Transform and Ordinary Differential Equation with Variable Coefficients. Advances in Theoretical and Applied Mathematics 6(1): 41-46.

Elzaki, T. M. (2011). The New Integral Transform “ELzaki Transform”, Global Journal of Pure and Applied Mathematics 7(1): 57–64.

Kim, H. (2014). The Shifted Data Problems by Using Transform of Derivative, Applied Mathematical Sciences 8(151): 7529-7534.

Schiff, J. L. (1988). The Laplace Transform: Theory and Applications. New York Berlin Heidelberg: Springer-Verlag. pp 1-58.

Sharjeel, S. and Barakzai, M. A. K. (2019). Elzaki transform applicaions for solution of problems arising in physics and finance. Science International 31(4): 631-636.

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